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FIRST-YEAR MATHEMATICS FOR SECONDARY SCHOOLS

First-Year Mathematics

For Secondary Schools

By

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PREFACE

This volume aims to present in teachable form an interweaving of the more concrete and the easier portions of the first courses in both algebra and geometry. It is intended as a text for students of the first year of secondary schools. The type of work begun here will be continued by a second-year text, to appear soon and to include not less than the rest of the subject-matters that make up the customary two years of required secondary work in algebra and plane geometry.

The present book places chief emphasis upon algebra, but weaves in a considerable body of related fundamental notions and principles of geometry, even advancing somewhat in one direction or another upon the most concrete, graphic, and practical aspects of elementary geometry. Geometrical treatments are at first intuitive, inductive, and experimental, followed in a few instances by the quasi-experimental methods of superposition.

A like informality as to method characterizes the first half of the algebra. The algebraic treatments feature distinctly: (1) the liberal use of inductions from arithmetic; (2) the early and persistent use of the equation, both as an instrument for problem-solving, and as a means of suggesting and of treating topics; (3) the early, varied, and systematic use of pictured and graphic modes of rendering algebraic truths vivid and appealing to beginners.

The algebra of the last half of the book is more formally deductive in character. Even as early as p. 26 the four process-axioms are given as simple statements of the four ways in which equations have been changed in solving arithmetical and mensurational problems. They are stated as laws for

using the equation, and are given only as much generality as the antecedent work justifies. Transformations of equations are then made on the basis of these laws and of others brought out from time to time as the work proceeds, until p. 216 is reached, where a fifth general axiom is added, and the five are here given final statement. These five axioms are henceforth made the *reasons* for all changes in equations. The root-axiom is added later when an occasion to use it arises. Thus the transition from the informal procedure of the earlier part of the algebra to the formal procedure of the later part is gradual, while throughout the latter half the reasoning becomes more and more highly deductive. This is in accord with what both classroom practice and a-priori reasoning show to be the natural order for secondary mathematicai education.

The subject-matter of the two years of work of this volume and its companion, *Second-Year Mathematics*, fully meets collegiate entrance requirements of one unit of algebra and one unit of plane geometry. There is gain in keeping the twofold point of view continually before the learner, habituating him to face his problems as mathematical problems rather than as algebra problems, or geometry problems. This requires him to keep both kinds of subject-matter longer before him, and brings to his help continually the reflex light that each subject throws upon the other. Thus the result must be a superior quality of collegiate preparation. The college teacher of pupils prepared on this plan, no less than the high-school teacher, will be gratified by the results the work of such students will show.

Colleges do not usually require, and they should not require, that the desired units of credit be made by the study of algebra and geometry as separate and distinct domains of mathematical truth. Much is lost educationally by this plan, and the authors are confident that college teachers will be

the first, the most numerous, and the warmest friends of the plan of these two books.

That such material as this book contains furnishes a better fitting for the mathematical needs of daily life than a year of formal algebra will hardly be denied. Accordingly it has no apologies to make for its appearance. It works an almost virgin soil in a field in which it elbows no competitors.

On the other hand, this book should not be looked upon in any sense as a protest against algebra as algebra, or geometry as geometry. Some teachers are succeeding measurably in making high-school algebra grip the motives of high-school boys and girls, and this is the main thing in mathematical teaching. For most teachers, however, it is much easier to motivate a school subject in a broad than in a narrow field. This is particularly true when in the narrow field the ideas are highly sophisticated and abstract, as is the case with school algebra. Greater freedom and more informality in getting at the beginner's field of ideas and fund of experiences are desired by many of our best mathematical teachers, and their number is increasing.

This text has been thoroughly seasoned through four years of classroom use in the University High School and purged by the ordeals of severe criticism. Six high-school teachers, with experience in high-school work ranging from five to twenty-five years, have participated in its authorship and in the practical tests of the classroom. The present form of the book is a thoroughgoing revision in the light of all this use and criticism. The conditions under which the material has been tried in every way are precisely the conditions of the public high school.

The marked improvement that has been wrought in the mathematical attitude and tone of the early classes of the University High School mainly through the agency of this text justifies strong claims for its suitability and worth for first-

year classes. Finality is, however, not claimed for it. It is claimed that this book will work *now*, and will work easily and effectively, under average conditions in the public high schools of the nation. Actual experience under such conditions proves beyond the power of a-priori argument to disturb, that without loss of mathematical solidity this book contributes in a superior degree to the much-neglected virtues of mathematical interest, earnestness, and spirit, and to an early and genuine faith in the worth of mathematical study.

The second-year text will make geometry the central theme and the algebra will take second rank. Algebraic matters will be related to kindred parts of geometry, and will, when needful, be given parallel or independent topical treatments.

The methodology of last treatments of this year will be increasingly deductive and logical, though even here austere ideals are not attempted. The secondary school can do profitably little more than rough out the bolder outlines of the sciences of algebra and geometry. The hold on the algebra learned the first year will be maintained, and the customary topics well along into simultaneous quadratics will be treated with customary fulness. Plane geometry will be completed, with careful regard to a smaller number than is customary of very well-worked-out propositions.

In conclusion the authors desire to render full acknowledgement to Dr. William B. Owen, dean of the University High School, for his interest and very substantial aid in furthering the practical side of this educational experiment. Without his assistance and continued interest, the task would have been difficult, if not impossible. He was among the first to recognize the importance of the function of the University High School as an experimental laboratory for secondary educational problems. Any success that may come to this book is due largely to him.

THE AUTHORS

CHAPTER I

GENERAL USES OF THE EQUATION

Problems and Exercises on the Balance

1. Problems in weighing are readily solved by the aid of the equation.

1. A bag of grain of unknown weight, w lb., together with an 8-lb. weight just balances an 18-lb. weight. How much does the bag of grain weigh?

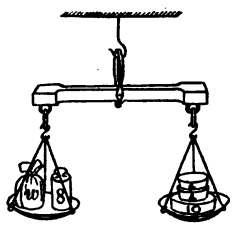


FIG. 1

The problem may be stated in an equation, thus:

$$w+8=18, \text{ find } w.$$

Suppose 8 lb. taken from each pan, giving

$$w=10.$$

The bag of grain weighs 10 pounds.

2. Two equal, but unknown, weights together with a 1-lb. weight just balance a 16-lb. and a 1-lb. weight together. How heavy is each unknown weight?

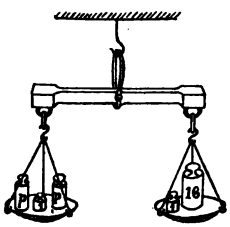


FIG. 2

In the form of an equation, the problem is stated:

$$2p+1=17; \text{ find } p.$$

Suppose 1 lb. to be taken from each pan giving

$$2p=16.$$

Then

$$p=8.$$

3. A man has only 2-lb., 4-lb., and 12-lb. weights. He finds that three bags of shot, of equal, but unknown, weights together with two 2-lb. weights on the left scale-pan and a

12-lb. and a 4-lb. weight on the right scale-pan, just balance. Find the weight of each bag of shot.

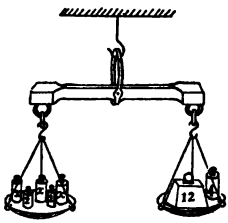


FIG. 3.—An equation is an expression of balance of values.

In the language of the equation, we may write

$$3x+4=16.$$

Taking 4 lb. from each pan, we have

$$3x=12.$$

Then

$$x=4.$$

Each bag of shot weighs 4 pounds.

4. A pail of berries of unknown weight, x lb., pressing downward, and a spring balance pulling upward and showing 4 lb. on the left side, and a weight of 12 lb. on the right side, just balance. How heavy is the pail of berries?

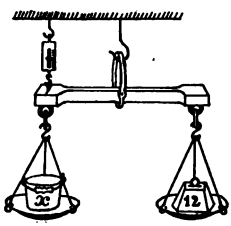


FIG. 4

In the language of the equation,

$$x-4=12.$$

Adding 4 lb. to both sides,

$$x=16.$$

The pail of berries weighs 16 pounds.

5. If there were three pails of berries of equal, but unknown, weights and a spring balance pulling upward and showing 3 lb. on the left side, just holding 42 lb. in balance on the right, find the weight of each pail of berries.

Using the equation,

$$3x-3=42.$$

Adding 3 lb. to both sides,

$$3x=45.$$

Then

$$x=15.$$

Each pail of berries weighs 15 pounds.

6. If a weight of $8x$ and an upward pull of 10 lb. just balance 70 lb. on the right, find the weight x.

Using the equation,

$$8x-10=70.$$

Adding 10 lb. to both sides,

$$8x=80.$$

Then

$$x=10.$$

The weight, x, equals 10 pounds.

2. Any equation, such as $5x+8=48$, $3x-5=10$, etc., may be regarded as an **expression of balance** between the numbers on the two sides of the sign (=) of equality.

Exercise I

Find the value of x in the following equations:

1. $x+3=5$	8. $4x-5=23$	15. $8x-7=57$
2. $2x-1=9$	9. $5x+7=28$	16. $8x-7=53$
3. $2x+6=16$	10. $5x-6=24$	17. $9x+12=93$
4. $3x+7=19$	11. $4x+10=22$	18. $9x-8=46$
5. $3x-4=11$	12. $4x-5=5$	19. $9x+8=116$
6. $3x+7=28$	13. $5x+17=62$	20. $9x-18=54$
7. $2x-11=21$	14. $6x-9=27$	21. $9x-7=74$.

Problems Using Algebraic Language

1. One-half of John's money is 9 cents. How much money has John?

Let x stand for the number of cents in John's money. Then, $\frac{1}{2}$ of $x=9$, and $x=2\times 9=18$. John has 18 cents.

"$\frac{1}{2}$ of x" is written $\frac{1}{2}x$, or $\frac{x}{2}$. The solution would then be:

$$\frac{1}{2}x=9,$$
$$x=18.$$

2. One-fourth of the number of acres in a field is 5 acres. How many acres are there in the field?

$$\frac{x}{4}=5.$$

Hence

$$x=20.$$

3. One-sixth of a certain number is 8. What is the number? Solve the problem by means of equations.

Exercise II

Give orally the number for which the letter stands in the following equations:

1. $\frac{a}{3}=9$
2. $\frac{c}{8}=7$
3. $\frac{s}{7}=6$
4. $\frac{m}{9}=8$
5. $\frac{r}{9}=7$
6. $\frac{t}{7}=9$
7. $\frac{x}{6}=7$
8. $\frac{x}{6}=9$
9. $\frac{y}{12}=8$
10. $\frac{z}{8}=12$
11. $\frac{s}{6}=12$
12. $3x=18$
13. $7x=49$
14. $8x=32$
15. $6x=48$
16. $5x=45$
17. $9x=72$
18. $8s=72$
19. $7x=63$
20. $9y=63$
21. $12z=84$
22. $11d=66$
23. $12r=72$
24. $9x=108$
25. $\frac{1}{9}y=7$
26. $\frac{1}{7}m=5$
27. $\frac{1}{12}s=6$
28. $12t=132$
29. $11x=121$
30. $63x=9$
31. $72y=8$
32. $56y=7$
33. $48r=8$
34. $35s=7$
35. $60d=50$
36. $1.2n=1.44$.

37. Helen had 8 cents and her father gave her 19 cents more. How many cents had she then?

38. A boy had 15 cents and was given c cents more. How many cents had he then?

3. The *sum* of 15 and c is written thus: $15+c$, or $c+15$. The first form is read "*fifteen plus c*" and the second form, "*c plus fifteen.*"

The *difference* of 15 subtracted from c is written $c-15$, and is read "*c minus fifteen.*"

1. A man had 22 acres and sold 8 acres. How many acres had he left?

2. A man had 22 acres and sold a acres. How many acres had he left?

3. A man had a acres and sold b acres. How many acres had he left?

4. A man had $5a$ acres and sold $2a$ acres. How many acres had he left?

Problems to be Solved by Arithmetic or Algebra

4. Many problems may be solved either by arithmetic, or by the use of the equation. When the solution of a problem is made by the use of the equation, it is commonly called an **algebraic solution.**

1. Divide a pole 20 ft. long into two parts so that one part shall be 4 times as long as the other.

ARITHMETIC SOLUTION

The shorter part is a certain length.
The longer part is four times this length.
The whole pole is then five times as long as the shorter part.
The pole is 20 ft. long.
The shorter part is $\frac{1}{5}$ of 20 ft., or 4 ft.
The longer part is $4 \cdot 4$ ft., or 16 ft.
Hence, the parts are 4 ft. and 16 ft. long.

ALGEBRAIC SOLUTION

Let x be the number of feet in the shorter part,
then $4x$ is the number of feet in the longer part,
and $x+4x$, or $5x=20$,

$$x=4,$$
$$4x=16.$$

Hence, the parts are 4 ft. and 16 ft. long.

2. A farmer wishes to inclose a rectangular pen with 80 ft.

of wire fencing. He wishes it to be three times as long as it is wide. How long shall he make each side?

ALGEBRAIC METHOD

Let x be the number of feet in the smaller side,
then $3x$ is the number of feet in the longer side,
and $x+3x$, or $4x$ is the number of feet half-way round the pen.

$$4x=40,$$
$$x=10,$$
$$3x=30.$$

Hence, the sides are 10 feet and 30 feet long.

3. A boy sold a certain number of newspapers on Monday, twice as many on Tuesday, 10 more on Wednesday than on Monday, and 24 on Thursday. He sold 94 in the four days. How many did he sell on each day?

4. A man divides up his 160-acre farm as follows: He takes a certain number of acres for lots, 4 times as much for pasture, 4 times as much for corn as for pasture, $\frac{1}{2}$ as much for wheat as for corn, and 15 acres for meadow. How many acres does he assign to each purpose?

5. A book dealer has in stock twice as many Readers as Arithmetics, four times as many Readers as Histories. In all he has 70 Readers, Arithmetics, and Histories. How many of each has he?

6. A may-pole 22 ft. high breaks into two pieces so that the top piece, hanging beside the lower piece, lacks 6 ft. of reaching the ground. How long is each piece?

7. A pony, a saddle, and a bridle together cost $120. The bridle costs $\frac{1}{5}$ as much as the saddle, and the pony costs $12 less than 12 times as much as the saddle. What was the cost of each?

8. A bicyclist rode a certain number of miles on Monday, $\frac{3}{4}$ as many miles on Tuesday, $\frac{5}{8}$ as many on Wednesday, $\frac{3}{4}$ as

many on Thursday, as many on Friday as on Monday, and 20 miles on Saturday. On the six days he rode 152 miles. How many miles did he ride each day?

9. James has 3 times as many cents as Charles, and 4 times as many as William. All together they have 57 cents. How many cents has each?

10. The area of a triangular piece of ground is 315 sq. rd. One side is 30 rods. How long is a fence at right angles to this side from the opposite corner? (Use principle below.)

PRINCIPLE.—*The area of a triangle is equal to ½ the product of its base and altitude.*

11. If this fence divides the side (30 rods) so that one part is twice as long as the other, what are the areas of the two lots?

12. If the fence divides the side (30 rods) so that one part is five times the other, what are the areas of the two lots?

13. A local train goes at the rate of 30 miles an hour. An express starts two hours later and goes at the rate of 50 miles an hour. In how many hours, and how far from the starting-point will the second train overtake the first?

14. If 75 dollars are to be divided between two persons so that one shall have 27 dollars more than the other, how much must each one have?

15. Three men, A, B, and C, wish to divide 1,584 shares of stock among themselves so that A shall have 25 more than B, and C shall have 50 more than B. How many shares must each receive?

16. At an election in the Freshman class of the University High School 84 votes were cast. Candidates A and B each received a certain number of votes; candidates C and D each received three times as many as candidate A; and candidate

E received four times as many as D, plus 4. How many votes did each candidate receive?

17. A regulation foot-ball field is $56\frac{2}{3}$ yards longer than it is wide and the sum of its length and width is $163\frac{1}{3}$ yards. Find its dimensions.

18. The manager of a high-school foot-ball team paid the manager of a visiting team \$63 for expenses. If the payment was made in \$2 and \$5 bills, the same number of each, how many bills of each kind were paid?

19. A line 40 ft. long is divided so that the longer part is 12 ft. more than 3 times as long as the shorter part. Find the length of each part. Draw a line 4 inches long to represent the 40 ft. line, and mark on it the lengths of the two parts.

20. Two men, A and B, wish to divide \$5,247 between them so that B shall receive \$324 more than twice as much as A. How much should each receive?

21. The length of a rectangle is 26 yd. greater than the width. If the perimeter (distance around) is 432 yd., find the dimensions.

22. Find two consecutive numbers whose sum is 203. (Suggestion: Let x be one number, and $x+1$ the other.)

23. Find three consecutive numbers whose sum is 474.

24. Find two consecutive odd numbers whose sum is 156. (Suggestion: Let x be one number and $x+2$ the other).

25. Find two consecutive even numbers whose sum is 378.

26. Find three consecutive even numbers whose sum is 372.

27. A college student spent during his Freshman year \$620 for tuition, room and board, and books. For tuition he spent four times as much as he did for books, plus \$20. For room

and board together he spent ten times as much as for books. How much did he spend for each item?

28. The tuition at the University High School is $150 a year. If this tuition is paid in twenty-dollar bills, and if the same number of five-dollar bills is received in change, how many twenty-dollar bills are paid?

29. Divide 341 into three parts so that the second part shall be five times the first part, and the third part shall be two times the first part plus 5.

30. A farmer wishes to lay out a field in the shape of a rectangle whose width is $\frac{3}{5}$ of its length, and whose perimeter is 320. How long and how wide must he make the field?

Exercise III

The following exercises test and review the addition and multiplication facts of arithmetic, and give exercise in algebraic equations. Find the value of the letter, x, doing all you can orally:

1. $2x+x=9$
2. $3x-x=8$
3. $2x+10=22$
4. $2x-7=13$
5. $7x-8=48$
6. $7x+1=64$
7. $7x+6=62$
8. $7x-9=40$
9. $7x+2x=27$
10. $7x-x=36$
11. $7x-2x=60$
12. $4x+5x=81$
13. $8x+2=66$
14. $8x-4=62$
15. $8x+9=81$
16. $8x-8=64$
17. $8x-2x=42$
18. $8x+x=81$
19. $8x+2x=70$
20. $8x-2x=72$
21. $3x+2x+x=15$
22. $4x+x-2x=21$
23. $6x+7x-5x=80$
24. $8x+4x-7x=60$
25. $5x+9x+2x=64$
26. $9x-4x-2x=36$
27. $11x-2x+3x=108$
28. $5x+6x+7x=36$
29. $7x+2x-8=46$
30. $7x-2x+6=41$

31. $12x-4x+x=90$
32. $3x-x+3x=20$
33. $6x+x-3x=18$
34. $7x-x+5x=77$
35. $12x+8x-5x=45$
36. $18x-9x+2x=66$
37. $19x-8x-3x=56$
38. $20x-8x+x=26$
39. $14x+4x-9x=63$
40. $12x+9x-3x=63$
41. $3x+7x+15x-2x+5=74$
42. $2x+7x-3x-6=24$
43. $\frac{1}{4}x=6$
44. $\frac{1}{2}x=3$
45. $\frac{2}{3}x=8$
46. $\frac{5}{6}x=20$
47. $x+\frac{1}{2}x=6$
48. $x-\frac{1}{2}x=7$
49. $\frac{3}{4}x+\frac{1}{2}x-\frac{1}{8}x=18$
50. $6x-\frac{2}{3}x-\frac{1}{4}x+7=129.$

Problems and Exercises

1. A man bought a certain number of ducks, three times as many turkeys, and five times as many chickens as turkeys. In all he bought seventy-six head of poultry. How many ducks did he buy? How many turkeys? How many chickens?

2. A thermometer showed a certain number of degrees rise one hour, and 2 degrees more than 3 times as many degrees rise the next hour. The rise for both hours was 14 degrees. Find the rise for each hour.

3. The rise of a mercury column in two hours amounted to 7 degrees, and the rise the second hour was 3 degrees less than 4 times the rise the first hour. Find the rise for each hour.

4. In two hours the total rise of the mercury was 3 degrees, and the rise the second hour was 5 degrees more than the change the first hour. Find the change for each hour. What do the results mean on the thermometer?

5. In two hours the total rise of the mercury was 2 degrees and the rise the second hour was 8 degrees more than twice the change the first hour. Find the change in the reading for each hour. Interpret the results by means of the thermometer.

The following exercises test and review the laws of arithmetical calculation, and give exercise in algebraic equations. Find the values of the letters, doing all you can orally:

6. $2x+1=6$
7. $3x-2=12$
8. $4y-3=12$
9. $5y+3=15$
10. $8s-7=53$
11. $8s-4s=26$
12. $9t+11=92$
13. $10t-11=104$
14. $12m+8=98$
15. $12t-6=74$
16. $12b+16=82$
17. $12k-9=43$
18. $12k+7=95$
19. $12x-15=30$
20. $16y+13=73$
21. $18y-12y=33$
22. $21z+15=102$
23. $25z-17=113$
24. $28x+14=158$
25. $28x-9=251$
26. $20x+2x-18x=22$
27. $17y-3y+16y=105$
28. $17s+7s-13s=88$
29. $16t+2t-13t=22\frac{1}{2}$
30. $3.4x-1.2x+4.8x=70$
31. $3.5x+7.6x-8.6x=15$
32. $5.8y-3.9y+12.6y=58$
33. $6s-3.5s+5.5s=68$
34. $7s-3.5s+16.2=54.7$
35. $6.82s+1.18s-3.5s=45$
36. $8x-4.5x+5.2x=87$
37. $4.5x-1.25x+12.7=38.7$
38. $16.5x+15.8-2.3x=186.2$
39. $6.15y-1.65y+7.8=57.3$
40. $8y+6.875+2y=46.875$
41. $z+5.37z-8.73=61.34$
42. $13t-8.75t+6.87=57.87$
43. $15x+3.73x-9.23=65.69$
44. $321x-109x+8x=22$
45. $404y-304y+12y=560$
46. $\frac{x}{2}+\frac{x}{4}=3$
47. $\frac{t}{3}-\frac{t}{6}=10$
48. $\frac{t}{5}+\frac{t}{3}=8$
49. $\frac{y}{4}-\frac{y}{7}=6$
50. $\frac{z}{2}-\frac{z}{3}=7$
51. $\frac{s}{4}-\frac{s}{6}=4$
52. $\frac{15}{a}=5$
53. $\frac{4}{x}=12$
54. $\frac{5}{z}=20$
55. $\frac{9}{r}=6.$

Problems on Percentage and Interest

1. Find the percentage of \$120 at 4%; at 6%; at $7\frac{1}{2}$%; at r%.

2. Find the percentage at 8% of \$25; of \$250; of \$1,527; of \$$b$.

3. Find the percentage at r% of \$20; of \$80; of \$$b$.

4. Calling p the percentage, b the base, and r the rate, show that

$$p=b\times\frac{r}{100}, \qquad (1)$$

and show that, by one of the laws of multiplication of fractions this may be written

$$p=\frac{b\times r}{100}. \qquad (2)$$

5. Read both (1) and (2) of problem 4 and translate them into words.

6. Find the interest on \$175 at 4% for 2 years; for 5 years; for $\frac{3}{4}$ of a year; for $2\frac{7}{8}$ years; for t years.

7. Find the interest on \$600 for 5 years at 3%; at 5%; at 8%; at $6\frac{1}{2}$%; at r%.

8. Find the interest on \$160 for t years at 6%; at $3\frac{1}{2}$%; at r%.

9. Find the interest at 6% for 3 years on \$200; on \$360; on \$756; on \$$p$.

10. Find the interest at 5% for t years on p dollars.

11. Find the interest i, at r% for t years on p dollars.

12. State in words the meaning of

$$i=p\times\frac{r}{100}\times t.$$

13. State in words the meaning of

$$i=\frac{p\times r\times t}{100}.$$

Important Conclusions

1. **An equation** is an **expression of balance** of values, or of numbers.

2. The equation is a convenient tool for problem-solving.

3. To solve a problem by the use of the equation, one equation is derived from another

(1) by *adding* the same number to both sides,

(2) by *subtracting* the same number from both sides,

(3) by *multiplying* both sides by the same number, or

(4) by *dividing* both sides by the same number.

4. Many problems may be more easily and simply solved by the equation than by arithmetical methods.

5. All the arithmetical processes of addition, subtraction, multiplication, and division, of both integers and fractions, must be well known in algebra.

6. Solving problems by the equation sometimes leads to subtracting larger from smaller numbers and necessitates using negative numbers, or "numbers below zero" on the thermometer scale.

7. Percentage and interest problems may be easily and quickly solved by the equation.

CHAPTER II

USES OF THE EQUATION WITH PERIMETERS AND AREAS

Indicating Arithmetical Operations Algebraically

1. A boy rides on his bicycle 8 mi. in one hour and 5 mi. the next hour; how far does he ride in the two hours?

NOTE.—Write the sum of 8 and 5, not 13, but $8+5$. The form $8+5$ is just as truly a sum as is 13. It will sometimes be desirable to refer to the form $8+5$ as the *indicated sum.*

2. If the boy rides a mi. the first, and 7 mi. the second hour, how far does he ride in the two hours?

3. A boy has m marbles and buys p more; how many has he then?

4. A boy had 18 marbles and lost 7; how many had he then?

NOTE.—Give the *indicated difference.*

5. A boy had m marbles and lost 8 of them; how many had he left?

6. A boy had m marbles and lost n of them; how many had he left?

7. Show the sums of these pairs of numbers and the differences, the first of the given numbers being the *minuend*, and the second, the *subtrahend:*

(1) x and 7	(5) s and t	(9) x and a
(2) a " 12	(6) 15 " n	(10) d " c
(3) y " 10	(7) r " s	(11) c " b
(4) x " y	(8) a " x	(12) t " m.

8. How many yards of cloth are 12 yd. and 10 yd.?

9. How many dozens of eggs are 8 doz. and 4 doz.?

10. How many 12's are 5 12's and 4 12's?

11. How many times 12 are 9×12 and 4×12?

12. How many half-dozens are 8 half-dozens and 6 half-dozens?

13. How many times 6 are 8×6 and 6×6?

14. How many times 3 are 5×3 and 7×3 and 10×3?

15. How many times x are 2 times x and 3 times x? x and x? x and 3 times x?

5. It is customary in algebra to write $x+x$, $x+x+x$, $x+x+x+x$, etc., thus, $2x$, $3x$, $4x$, etc., and to read them "two x," "three x," "four x," and so forth.

But x times x, x times x times x, and x times x times x times x are written x^2, x^3, x^4, etc., and are read, "x square," "x cube," "x fourth power," etc.

x square may be read "x 2d power" or "x 2d;" x cube may be read "x 3d power," or "x 3d;" and "x fourth power" may be called "x 4th," etc.

The dot is often used instead of the times sign to indicate multiplication. Thus, $x\times x\times x$ may be written $x\cdot x\cdot x$, meaning x times x times x.

6. In $2x$, $3x$, $4x$, ax, etc., the 2, 3, 4, a, etc., are called **coefficients** of x.

7. In x^2, x^3, x^4, x^n, etc., the 2, 3, 4, n, etc., are called **exponents** of x.

What would be the written form of "x 5th?" "x 6th?" "x 7th?" "x 10th?" "x nth?" What would be the meaning of each of these forms?

8. Notice that $4x$ means $4\cdot x$, or that x is to be used *as an addend* 4 times, while x^4 means $x\cdot x\cdot x\cdot x$, or that x is to be used *as a factor* 4 times, and similarly, for the other forms, as $3x$ and x^3, and $5x$ and x^5, etc.

Letting $x=6$, give the values of

1. $2x$	4. x^3	7. $5x$	10. $3x^2$
2. x^2	5. $4x$	8. x^5	11. $5x^3$
3. $3x$	6. x^4	9. $2x^2$	12. $2x^4$.

13. Describe briefly the meaning of *factor* and of *addend* as they are used in arithmetic.

Expressing Perimeters and Areas

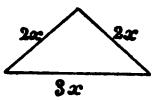


FIG. 5

1. What is the sum of the sides of a triangle (Fig. 5) whose sides are $2x$ ft., $2x$ ft., and $3x$ ft. long? Express the sum as a certain number of times x.

2. What is the sum of the three sides $2a$, $5a$, and $6a$ of a triangle?

3. A lot has the form of an equal-sided (equilateral) triangle (Fig. 6), each side being x rd. long. How many rods of fence will be needed to inclose it?

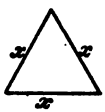


FIG. 6

9. Any figure whose sides are all equal is called an *equilateral* figure; as equilateral triangle, equilateral pentagon, etc.

The sum of all the sides of any closed figure is called the *perimeter* of the figure.

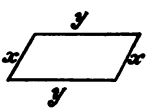


FIG. 7

1. In Fig. 7, called a parallelogram, what part of the *perimeter* does $x+y$ equal? What is the whole perimeter?

2. Show (1) that the perimeter, p, of a square whose side is 8 is given by $p=4 \cdot 8$ and (2) that the area, A, is given by $A=8^2$.

3. Show by equations the perimeter, p, and the area, A, of a square whose side is

(1) 12 (4) a (7) $10+2$
(2) s (5) y (8) $b+7$
(3) x (6) $2b$ (9) $c+d$.

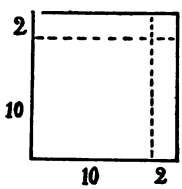


FIG. 8

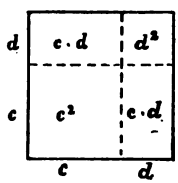


FIG. 9

4. What is the perimeter of a square of which the area is 16 sq. ft.? a^2 sq. ft.? $4a^2$ sq. ft.? x^2 sq. ft.?

5. What is the area of a square whose perimeter is 20 ft.? $4x$ ft.? $8a$ ft.?

6. Draw figures to illustrate your solutions of 5.

7. The dimensions of a rectangle are a and b, what is the perimeter? The half-perimeter? The sum of a pair of opposite sides? The sum of the other pair?

8. What is the area of the rectangle of problem 7?

10. Numbers denoted by letters are *literal numbers*. The product of two different literal numbers, as x and y, is shown by writing the letters, or factors, side by side, as xy, with no sign between. We are familiar with the form $x \times y$ from arithmetic. The form xy is most used in algebra. It is often convenient to use the form $x \cdot y$.

1. Recalling that the exponent of a number shows how many times the number is to be used as a factor, give the products of the following pairs of factors:

(1) $4^2 \cdot 4^2$	(5) $a^2 \cdot a$	(9) $b^2 \cdot b^5$
(2) $8 \cdot 8^3$	(6) $a^3 \cdot a^3$	(10) $c \cdot c^8$
(3) $10^2 \cdot 10^4$	(7) $a \cdot a^5$	(11) $c^3 \cdot c^6$
(4) $12^3 \cdot 12^4$	(8) $a^4 \cdot a^7$	(12) $x^3 \cdot x^8$.

2. Show from Fig. 10, (1) that the perimeter, p, of a square of $3x$ is given by $p = 12x$, and (2) that the area, A, is given by $A = 9x^2$.

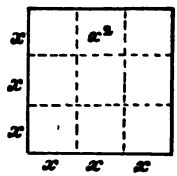


FIG. 10

3. Express by an equation the area, A, of a rectangle that is 8 in. long and 5 in. wide; of the same length and 4 in. wide; $3\frac{1}{2}$ in. wide; $6\frac{1}{4}$ in. wide.

4. Express by an equation the area, A, of a rectangle 12 in. long and of the following widths: 6 in.; $8\frac{1}{4}$ in.; $9\frac{1}{3}$ in.; $10\frac{3}{4}$ in.; x in.; y in.

5. Express by equations the areas of rectangles l in. long and of the following widths: 12 in.; 9 in.; h in.; n in.; x in.; a in.

6. Express by equations the areas of rectangles of width w and of the following lengths: 8; 10; $12\frac{1}{2}$; x; a; l; b; z.

7. Express by equations the products of the following pairs of factors:

(1) a by x	(5) b by b^3	(9) x^3 by x
(2) b by c	(6) a^2 by a	(10) a by x^2
(3) b by b	(7) a^2 by a^3	(11) a^2 by x
(4) a by a^2	(8) x by x^3	(12) g by t^2.

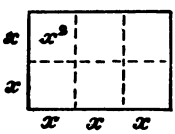


FIG. 11

8. Show from Fig. 11, that the area, A, of a rectangle $2x$ by $3x$ is given by $A=6x^2$.

9. Show that the area, A, of a rectangle $3a$ by $5a$ is given by $A=15a^2$.

10. Express by an equation the area, A, of parallelograms of altitude a, and of the following lengths: 10; $12\frac{1}{2}$; 16.8; b; c; x; $2n$; $5z$.

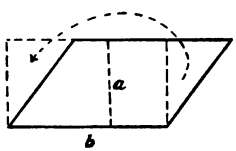


FIG. 12

11. Express by an equation the area, A, of triangles of altitude a, and of the bases, 12; $18\frac{1}{3}$; 20.25; b; d; x; $2y$; 42; $5c$.

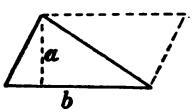


FIG. 13

12. Show that the area, A, of the trapezoid, in the accompanying Fig. 14, is given by $A=4\cdot\frac{6+8}{2}$.

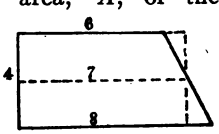


FIG. 14

13. Show that the area, A, of the

trapezoid, in the accompanying Fig. 15, is given by $A=3\cdot\frac{9+7}{2}$.

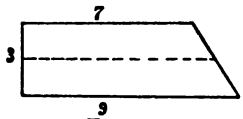


FIG. 15

14. Show that the line EF $=\frac{1}{2}(b+c)$, meaning $\frac{1}{2}$ of the sum of b and c of Fig. 16.

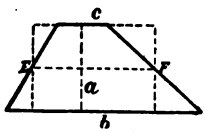


FIG. 16

15. Show from Fig. 16 that the area, A, of the trapezoid is given by $A=a\cdot\frac{b+c}{2}=\frac{a(b+c)}{2}$.

16. Express by an equation the area, A, of a rectangle of dimensions

(1) 6 and $x+3$ (3) a and $x-2$
(2) x and $a+5$ (4) $a+2$ and $x+1$.

17. Indicate the area of a rectangle of dimensions $a+b$ and $x+y$.

NOTE.—The product of $m+n$ and $c+d$ is written $(m+n)(c+d)$.

18. Express in terms of its base and altitude the area of the rectangle (1) of Fig. 17; of (2); of (3); of (4).

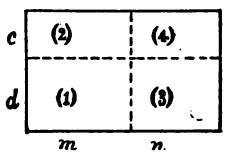


FIG. 17

19. How then may you express $(m+n)(c+d)$ by using the truth that any whole equals the sum of all its parts?

20. State in words the value of

(1) $(x+y)(a+b)$ (3) $(a+x)(b+y)$
(2) $(x+y)(m+n)$ (4) $(r+s)(a+x)$.

21. Show by a figure the value of

(1) $(a+b)(a+b)$ or $(a+b)^2$ (3) $(x+y)^2$
(2) $(c+d)^2$ (4) $(m+n)^2$.

22. Each of the following expressions is the product of what two equal numbers?

(1) $a^2+2ax+x^2$	(3) $k^2+2kb+b^2$	(5) x^2+6x+9
(2) $b^2+2bc+c^2$	(4) $s^2+2st+t^2$	(6) $c^2+8c+16$.

23. The base of a rectangle is 8 yd. and the area is 40 sq. yd. What is the altitude, a?

24. What is the altitude, a, of a rectangle having—

a base = 8 in. and an area = 32 sq. in.?
" = 8 in. " " = 16 sq. in.?
" = 8 in. " " = 12 sq. in.?
" = 5 ft. " " = $7\frac{1}{2}$ sq. ft.?

25. Show by an equation the base, b, of a rectangle having—

an altitude = 9 ft. and an area = 27 sq. ft.
" = 9 ft. " " = 18 sq. ft.
" = 9 ft. " " = 15 sq. ft.
" = 9 ft. " " = 12 sq. ft.
" = 9 ft. " " = A sq. ft.

26. Show by an equation the other dimension of a triangle having—

a base = 6 ft. and an area = 24 sq. ft.
" = 6 ft. " " = 12 sq. ft.
" = 6 ft. " " = 9 sq. ft.
" = 6 ft. " " = 3 sq. ft.

an altitude = 4 yd. and an area = 16 sq. yd.
" = 4 yd. " " = 8 sq. yd.
" = 4 yd. " " = 4 sq. yd.
" = a ft. " " = a sq. ft.
" = h rd. " " = hl sq. ft.
" = b in. " " = ab sq. in.
" = b in. " " = by sq. in.
" = c in. " " = a sq. in.
" = a in. " " = b sq. in.

11. The **quotient** of x divided by y is written $\frac{x}{y}$, or $x \div y$ and is read "x divided by y." $\frac{x}{y}$ is also read "x over y."

Show by an equation the quotient, q, of the first of these numbers divided by the second:

1. m	and	n	5. $a+b$	and	c	9. a^2-b^2	and	$a-b$
2. a	"	b	6. a	"	$x+y$	10. x^2	"	$a+b$
3. b	"	a	7. $a+b$	"	$c+d$	11. $(a+b)^2$	"	$a+b$
4. $4x$	"	$3y$	8. a^2-b^2*	"	$a+b$	12. $(a-b)^2$	"	$a-b$.

*NOTE. a^2-b^2 is read "a square minus b square;" $(a-b)^2$ is read "the square of $a-b$," and $(a+b)^2$ is read "the square of $a+b$."

Definitions

12. In an expression like $3xy+7ax-9b-4$, the numbers $3xy$, $7ax$, $9b$, and 4, that are connected by *addition* or *subtraction*, are called **terms** of the expression.

13. *A one-term expression*, like 8, $7x$, abc^2, etc., is called a **monomial.**

14. The **coefficient** of any factor, or number in a term is the *product* of all the other factors of the term. Thus, in $4axy$, the coefficient of axy is 4, of xy is $4a$, of ax is $4y$; in $\frac{bc}{3}$ the coefficient of c is $\frac{b}{3}$, of b is $\frac{c}{3}$, etc.

When the coefficient of *the term* is spoken of the arithmetical factor is usually meant. Thus, it is common to say the coefficient of the term $4axy$ is 4; of $\frac{bc}{3}$ is $\frac{1}{3}$, etc. If no coefficient is written as in a, x^4, 1 is understood, as though $1a$, $1x^4$ were written.

15. The **exponent** of a number is the small figure or letter written to the right and a little above the number symbol, to

denote *the number of equal factors* in a product. Thus, in 6^3, meaning the product $6 \times 6 \times 6$, the 3 is the exponent. The number 6, itself, is called the **base**, and the product 6^3 is the **power.** Thus, 216 ($=6^3$) is the 3d power of 6, because using 6 as a *factor* 3 times gives 216, and so with other numbers.

When no exponent is written, as in ax, the exponent is understood to be 1 for each letter, as though the number were written $a^1 x^1$.

16. A number as $4ax+10$, denoted by *an expression of two terms* is a **binomial;** *a three-term expression* is a **trinomial.**

17. Number expressions consisting of two, three, four, etc., terms, are called **polynomials.**

The area of a rectangle whose altitude is 5 and whose base is the sum of two lines, one a inches and the other b inches long, may be written thus: $5(a+b)$. If the altitude were 7 and the base equal to the difference, $a-b$, the area would be written $7(a-b)$.

18. The sign, ( ), called the **parenthesis,** means that the terms within, as the a and the b in **17**, are first to be added, or subtracted, and then the sum or difference is to be multiplied.

Give the meaning of the following:

1. a^3	4. $8(40+2)$	7. $16(a+b)$
2. $4a^2b$	5. $9(60-3)$	8. $20(x-y+1)$
3. $6(10-3)$	6. $7(80+1)$	9. $25(x-a)$.

Finding Values of Expressions

Of what kind of figures may the following equations express the perimeters, p?

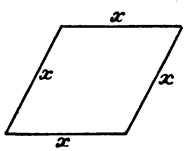


FIG. 18

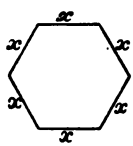


FIG. 19

1. $p=3x$
2. $p=4x$
3. $p=5x$
4. $p=6x$
5. $p=7x$
6. $p=8x$
7. $p=9x$
8. $p=10x$
9. $p=12x$
10. $p=20x$
11. $p=nx$
12. $p=ax$.

13. Give the coefficient of x in the second members of problems 1 to 12.

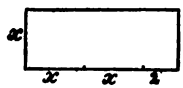


FIG. 20

Show, by a sketch, figures whose perimeters, p, are given by

14. $p=2(2x+2)$
15. $p=2(3x+2)$
16. $p=6x+4$
17. $p=4x+16$
18. $p=6x-12$
19. $p=12x-24$
20. $p=3x+3y$
21. $p=4x+2y$
22. $p=5x+3y$.

23. Find the values of the perimeters in problems 1 to 12 for $x=2$ in.; for $x=3$ ft.; for $x=6$ yd.

24. Find the values of the perimeters, p, of problems 14 to 22, for $x=5$ and $y=2$; for $x=12$ and $y=4$; for $x=a$ and $y=b$.

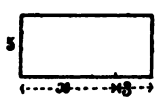


FIG. 21

Of what plane figures do the following equations give the areas, A?

25. $A=x^2$
26. $A=3ax$
27. $A=2(x+1)$
28. $A=5(x+3)$
29. $A=x(a+2)$
30. $A=a(x+4)$
31. $A=x(y-2)$
32. $A=y(x+7)$
33. $A=\frac{3(x+1)}{2}$
34. $A=\frac{4(a+b)}{2}$
35. $A=a\left(\frac{b+c}{2}\right)$
36. $A=x^2-4x$.

37. Find the value of A in problems 25 to 36 for $x=3$, $y=2$, $a=4$, $b=1$, $c=1$; for $x=5$, $y=3$, $a=6$, $b=2$, $c=4$.

19. An expression like $(a+5)(x-y)$ means that a and 5 are first to be added, then y is to be subtracted from x, and the sum, $a+5$, is to be multiplied by the difference, $x-y$.

Give the meanings of the following:

1. $(x+y)(a+10)$
2. $(a+b)(c+d)$
3. $(x+y)(a-b)$
4. $(a-b)(c-d)$.

Sketch and show dimensions of rectangles whose areas are given by

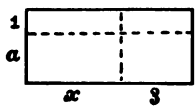


FIG. 22

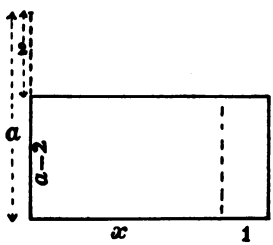


FIG. 23

5. $A=(a+1)(x+3)$
6. $A=(a+5)(x+4)$
7. $A=(a-2)(x+1)$
8. $A=(a-3)(x+6)$
9. $A=(a-1)(b+1)$
10. $A=(a-2)(b+2)$
11. $A=(a-1)(b-1)$
12. $A=(a-1)(a+1)$
13. $A=(a+1)(a+1)$
14. $A=(a+2)(a+4)$
15. $A=(a-4)(a+4)$
16. $A=(a+b)(a+b)$.

20. The signs, $\sqrt{}$ and $\sqrt[3]{}$, called **radical signs,** have the same meaning as in arithmetic.

Give the values of the following for $a=3$, $b=3$, $c=1$, $t=8$, $x=10$:

1. $(a+b)(c+1)$
2. $8(a+x)$
3. $a(b+c)$
4. $t(ab-c)$
5. $a(x+b-c)$
6. $3x(ab+8-5c)$
7. $(a+c)(x-t)$
8. $a+b-\sqrt[3]{t}$
9. $\sqrt{a^2}$
10. $\sqrt{ab}$
11. $\sqrt{bt}$
12. $\sqrt[3]{\frac{t}{c}}$
13. x^b
14. $25x^3$.

The Use of the Fundamental Laws in Equations

1. The width of a rectangle is $x-3$, the length x, and the perimeter is 66 yards. Find the width and the length in yards.

The perimeter being denoted by both 66 yards and $4x-6$, we write the equation

$$4x-6=66. \quad (1)$$

Adding 6 to both sides

$$4x=72. \quad (2)$$

Dividing both sides by 4

$$x=18. \quad (3)$$

Subtracting 3 from both sides

$$x-3=15. \quad (4)$$

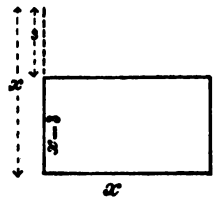


FIG. 24

The width is 15 yards and the length 18 yards.

CHECK: $15+18+15+18=66$, or from equation (1) $4 \cdot 18-6=72-6=66$.

2. The width of a rectangle is x ft., the length $x+12$ ft., and the perimeter is 144 ft. Find the width and length.

Both 144 and $4x+24$ denoting the perimeter, we write

$$4x+24=144. \quad (1)$$

Subtracting 24

$$4x=120. \quad (2)$$

Dividing by 4

$$x=30, \text{ the width.} \quad (3)$$

Adding 12

$$x+12=42, \text{ the length.} \quad (4)$$

CHECK: $30+42+30+42=144$, or from equation (1) $4x+24=4 \cdot 30+24=120+24=144$.

3. One-fifth of the perimeter of a field plus $3\frac{1}{2}$ rods is 10 rods. Find the perimeter.

Call p the perimeter, then

$$\frac{p}{5}+3\tfrac{1}{2}=10. \quad (1)$$

Multiply both sides by 10

$$2p+35=100. \quad (2)$$

Subtract 35

$$2p=65. \quad (3)$$

Divided by 2

$$p=32\tfrac{1}{2}. \quad (4)$$

CHECK: $\frac{32\frac{1}{2}}{5}+3\frac{1}{2}=6\frac{1}{2}+3\frac{1}{2}=10.$

4. In the solution of problem 1 show where each of the following laws is used:

(1) If the same number be added to equal numbers, the sums are equal;

(2) If equal numbers be divided by the same number, the quotients are equal;

(3) If the same number be subtracted from equal numbers, the remainders are equal.

5. Show where the laws of problem 4 are used in the solution of problem 2.

6. In the solution of problem 3, show where the law is used: "if equal numbers be multiplied by the same number the products are equal.

7. In the same solution show where the following are used:

"If the same number be subtracted from equals the remainders are equal."

"If equal numbers be divided by the same number the quotients are equal."

21. The four laws thus far used are

I. *If the same number be added to equals the sums are equal,*

II. *If the same number be subtracted from equals the remainders are equal,*

III. *If equal numbers be multiplied by the same number, the products are equal,*

IV. *If equal numbers be divided by the same number, the quotients are equal.*

Solve the following, using the equation and showing which fundamental law is used in each step.

1. The perimeter of an equilateral triangle is 27 rods. Find a side.

2. The perimeter of an equilateral quadrilateral is 48 ft. Find a side.

3. The perimeter of an equilateral hexagon (6-side) is 186. Find a side.

4. The perimeter of an equilateral decagon (10-side) is 255. Find a side.

5. The perimeter of an equilateral dodecagon (12-side) is 294. Find a side.

6. The area of a rectangle is 96 yards, the base is $x+8$ and the altitude is 8. Find the base. Find x.

7. The altitudes, bases, and areas of parallelograms are as given. Find the value of the unknown dimension in arithmetical numbers.

Altitudes	Bases	Areas
$x-3$	4	12
$3x+2$	8	56
10	$3x+2$	110
16	$5x-7$	288
$2x+1$	15	105
25	$60a+40$	2,500
$28y-17$	30	840

Application of the Fundamental Laws

22. The four fundamental laws stated above may be referred to thus:

I is the **addition-law**, **II**, the **subtraction-law**, **III**, the **multiplication-law**, and **IV**, the **division-law**. These laws are also called **axioms**.

EXERCISE IV

Solve the following equations, i. e., find the number for which the letter stands, and show where each of the four laws just stated is used:

1. $x-5=12$
2. $2x-6=18$
3. $3x+1=19$
4. $3x-1=20$
5. $3x+4=25$
6. $3x-4=14$
7. $4x+1=21$
8. $4x-6=18$
9. $4x+2=26$
10. $4x-16=32$
11. $12x=72$
12. $11x=90$
13. $\frac{3}{2}x-2=10$
14. $\frac{2}{3}x+14=20$
15. $\frac{4}{5}x-10=2$

8. An *equal-sided* figure is an **equilateral** figure.

9. How to denote perimeters and areas of geometrical figures by algebraic expressions, and equations.

10. How to find the product of binomials by monomials, and of two binomial sums, both algebraically and geometrically.

11. A single number, or a product or quotient of numbers is a **term.**

12. A *one-term number* is a **monomial**; a *two-term number* is a **binomial**; a *three-term number* is a **trinomial**; and a number expressed by *any number of terms* is a **polynomial.**

13. The **coefficient** of any factor in a term is the *product* of all other factors of the term.

14. The coefficient *of a term* usually means the *arithmetical* factor of the term.

15. The **exponent** of a power is the number that shows how many times the *base* is a *factor* in the *power.*

16. The value of a letter is the number for which the letter stands.

17. To find the value of an expression means to substitute values for the letters, and to reduce the result of all operations indicated to the simplest form.

18. Changes in an equation can be made only in accordance with a few **fundamental laws,** called **axioms.**

CHAPTER III

THE EQUATION APPLIED TO ANGLES

The Measurement of Angles

23. What part of a complete turn does the hand of a clock or watch make (Fig. 25) when it rotates (turns) about the center post from 12 to 3? 12 to 6? 12 to 9? 12 to 4? 12 to 8? 12 to 2? 12 to 10?

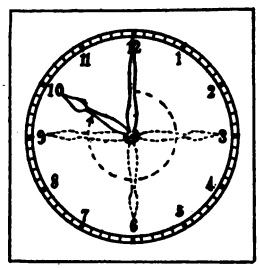


FIG. 25

24. If a line, OA (Figures 26, 27, 28, 29), be imagined to rotate (turn) in a plane about a fixed point, O, in the direction indicated by the arrow-heads, until it reaches the position O B, it is said to turn through the **angle** x (also

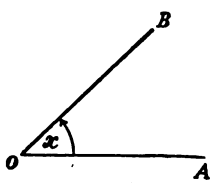


FIG. 26

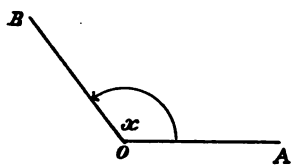


FIG. 27

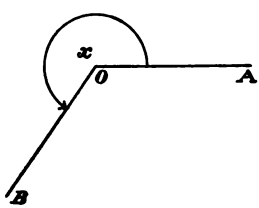


FIG. 28

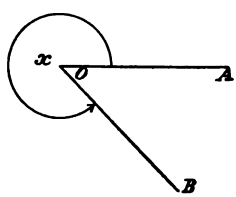


FIG. 29

called the **angle** A O B). In Fig. 30 the rotating line has made a complete turn about the point O.

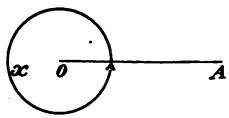


FIG. 30

The word "angle" means "the amount of turning" of a rotating line. The original meaning of the word was "corner," from the Latin *angulus*.

Draw an angle made by a line which has rotated $\frac{1}{4}$ of a complete turn; $\frac{1}{2}$ of a complete turn; $\frac{3}{4}$ of a complete turn; $1\frac{1}{4}$ turns; $1\frac{3}{4}$ turns.

25. If a line, rotating in a plane about one of its points, makes $\frac{1}{4}$ of a complete turn, the angle it makes is called a **right angle** (Fig. 31); if the line makes a half-turn the angle is called a **straight angle** (Fig. 32).

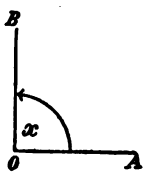


FIG. 31

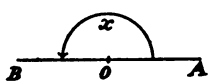


FIG. 32

1. Draw an angle equal
   (1) to two right angles
   (2) to three right angles
   (3) to four right angles
   (4) to five right angles
   (5) to two and a half right angles
   (6) to two straight angles
   (7) to one and a half straight angles
   (8) to two and a half straight angles
   (9) to three-fourths of a straight angle
   (10) to one and three-fourths straight angles.

2. How many right angles are there in an angle made by

a complete turn of a rotating line? By a half turn? By 7 turns? By $1\frac{1}{4}$ turns? By t turns? By $\frac{t}{2}$ turns? By $\frac{3t}{4}$ turns? By $\frac{5t+7}{2}$ turns?

3. How many straight angles are there in an angle made by a complete turn of a rotating line? By $1\frac{1}{2}$ turns? By $\frac{3}{4}$ of a turn? By $\frac{1}{4}$ of a turn? By 9 turns? By $3t$ turns? By $\frac{7t}{4}$ turns? By $3(2t-5)$ turns?

4. How many right angles are there in a straight angle? In 5 straight angles? In $7\frac{1}{2}$ straight angles? In s straight angles? In $\frac{s}{6}$ straight angles? In $\frac{13s}{8}$ straight angles? In $9s+8$ straight angles? In $3s-2$ straight angles?

5. How many straight angles are there in 4 right angles? In 2 right angles? In 3 right angles? In 7 right angles? In r right angles? In $\frac{4r}{3}$ right angles? In $\frac{16r}{5}$ right angles? In $6r+8$ right angles? In $4r-6$ right angles?

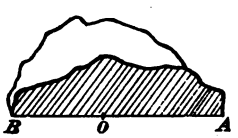


FIG. 33

6. Fold a crease (A O B, Fig. 33) in a piece of paper. Fold again so that the edge, O A, falls along the edge, O B. Unfold the paper. The two creases, A O B and C O D (Fig. 34), form four angles, x, y, z, w. Show that these angles are equal. Are they right angles? Why?

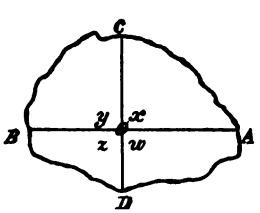


FIG. 34

26. On the **protractor** (angle measurer) shown in Fig. 35, the right angle, XOY, is divided into 90 equal angles. Each of these angles is a **degree.**

35 degrees is written briefly, $35°$. The symbol for angle is $\angle$; for angles, is $\angle s$.

1. How many degrees are there in $\angle XOA$ (Fig. 35)? In $\angle XOB$? In $\angle AOB$? In $\angle XOC$? In $\angle AOC$?

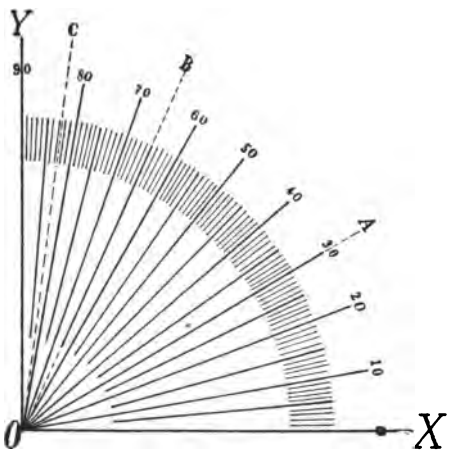


FIG. 35

2. Express by an equation the number of degrees in $\angle XOA + \angle AOB$ (Fig. 35); in $\angle AOB + \angle BOY$; in $\angle XOB - \angle AOB$; in $\angle XOB - \angle XOA$.

3. How many degrees are there in 2 straight angles? In 4 right angles? In $\frac{2}{3}$ of a right angle? In $\frac{7}{9}$ of a right angle? In $\frac{1}{5}$ of a right angle? In r right angles? In $\frac{7r}{18}$ right angles? In $r+7$ right angles? In $\frac{r+3}{2}$ right angles?

27. The protractor is commonly a semicircle (Fig. 36). If a straight line were drawn from each mark on the circular

rim to the center, O, 180 equal angles would be formed at O each of which would be an angle of one degree (1°).

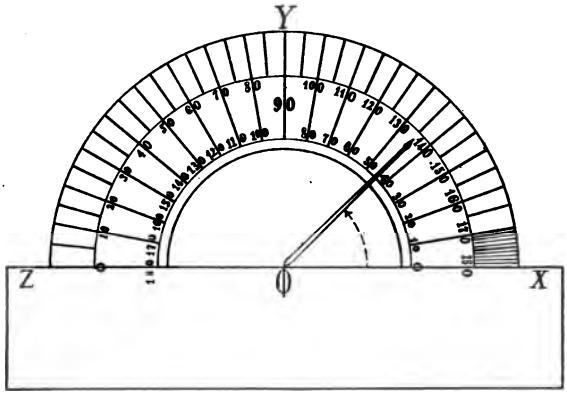


FIG. 36

1. Suppose a pointer to rotate about the fixed point, O (Fig. 36), in the direction indicated by the arrow. In rotating from the position O X to the position O Z through how many right angles would it turn? Through how many straight angles? Through how many degrees?

2. Find the number of degrees and the number of right angles in each of the following angles of Fig. 37: X O A; X O B; X O Y; X O C; X O D; A O B; A O Y; B O C; B O D; C O Z; A O Z; X O Z.

3. Read the angle in Fig. 37 which is the sum of angles X O A and A O B; of angles A O B and B O C; of angles X O B and B O Y.

4. Angle X O B is the sum of what two angles of Fig. 37? Answer by an equation.

5. Show by an equation that the sum of two angles of Fig. 37 equals ∠A O C; ∠X O C.

6. Show by an equation that the sum of three angles of Fig. 37 equals ∠X O C; ∠A O C.

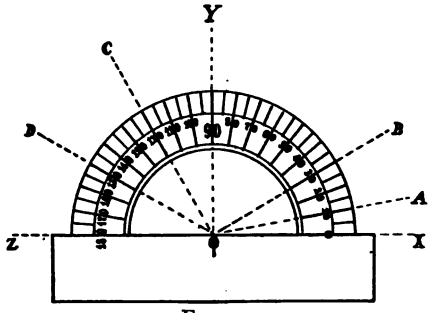


FIG. 37

7. If ∠X O A is taken from ∠X O B (Fig. 37), what angle is left? Answer by an equation.

8. Show by an equation the angle that is the difference between ∠X O Y and ∠X O A (Fig. 37); ∠A O Y and ∠A O B; ∠X O Y and ∠B O Y.

9. From a point, O, on a straight line X Y (Fig. 38) draw three lines as in the figure. Find the number of degrees in each angle, first, by estimating, then by measuring with the protractor. Tabulate results as in Fig. 39, and find the sum.

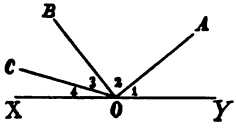


FIG. 38

Angle	Estimated	Measured
1	40°	39°
2	90°	
3		
4		
Sum		

FIG. 39

10. What should be the sum of all the angles about a point, on one side of a straight line? Check the sum in exercise 9.

NOTE.—To *check* results means to show that they are correct. Students should habitually check results, because correct results show that the *work* is probably correct.

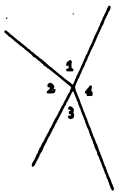


FIG 40

11. Draw four lines from a point (Fig. 40). Find the number of degrees in each angle, first, by estimating, then by measuring with a protractor. Tabulate results as in exercise 9, and check.

12. What is the sum of the angles that just fill the plane about a point?

The Sum of the Angles about a Point

1. Find the number of degrees in each angle of Fig. 41.

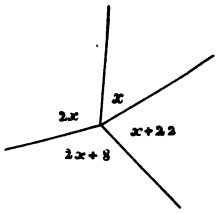


FIG. 41

We may write

$$x+2x+2x+8+x+22=360. \quad \text{Why?} \qquad (1)$$

This equation may be written

$$x+2x+2x+x+8+22=360. \quad \text{Why?} \qquad (2)$$

Combining like terms	$6x+30=360.$		(3)
Subtracting 30	$6x=330.$	What axiom is used here?	(4)
Dividing by 6	$x=55.$	What axiom?	(5)
Whence	$2x=110.$	Why?	(6)
	$2x+8=118.$	Why?	(7)
	$x+22=77.$	Why?	(8)

CHECK: $55+110+118+77=360.$

2. With a protractor measure the angles of Fig. 41 to see if the figure in the book is accurately drawn.

3. To obtain the second equation, problem 1, from the first equation, the order of the terms was changed on the left side of the first equation. Why may this be done?

4. Perform the following additions and subtractions:

(1) $8+7+3+6$	(5) $8+7-3+6$	(9) $a+7$
(2) $8+3+6+7$	(6) $8+7+6-3$	(10) $7+a$
(3) $7+6+8+3$	(7) $7-3+8+6$	(11) $3a+7+a$
(4) $6+8+3+7$	(8) $6-3+8+7$	(12) $3a+a+7$.

5. In problem 4 compare the results of (1), (2), (3), and (4); of (5), (6), (7), and (8); of (9) and (10); of (11) and (12).

6. In the following problems perform the additions and subtractions in different orders. Compare the results of each problem for the different orders, each term as it changes position carrying with it the sign next before it:

(1) $12+7-4+6$	(3) $7x+5-3x+2$
(2) $7x+5-3x$	(4) $7x+5-3x-2$.

7. What conclusion do you draw from problems 4, 5, and 6?

28. The *order of terms* of an expression may be changed, without changing the *value* of the expression, provided each term as it changes position carries with it the sign next before it.

1. Find the number of degrees in each angle of Fig. 42, writing the solution as in the foregoing problem 1.

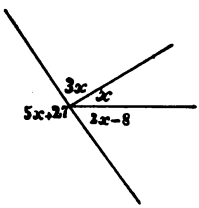


FIG. 42

All the angular space about a point in a plane is divided into angles represented by the following expressions; find x and each angle in degrees. With protractor draw figures for 2, 3, and 4:

2. $2x$, x, $4x+40$, $180-3x$
3. x, $36+5x$, $3x-9$
4. $3x$, $5x$, $5x+45$, $27-x$
5. $15x+16\frac{1}{2}$, $37\frac{1}{2}-2x$, $8x-9$
6. $3x$, $5x+26\frac{2}{5}$, $2x$, $9x+143\frac{3}{5}$
7. $7x+24$, $14x+53\frac{3}{8}$, $120\frac{5}{8}-3x$.

All the angular space about a point in a plane, on one side of a straight line, is divided into angles represented by the following expressions. Find x and each angle in degrees. Draw figures for 8, 9, and 10:

8. $x,\ 5x,\ 7x-2$
9. $9x,\ x,\ 37-2x,\ 5x-26$
10. $8x,\ 48-3x,\ 5x-22,\ 4x-14$
11. $25\frac{2}{3}+5x,\ 8x+8\frac{5}{6},\ 6x,\ 9\frac{1}{2}-2x$
12. $3x,\ 2(x+9),\ x,\ 42-x$
13. $2x,\ 2(x+10),\ x-18,\ 3(36-x)$
14. $3(x-3),\ x+33,\ 2(41-x)$
15. $2.8x+39.33,\ 1.2x-32.09,\ x+7.16$
16. $6.93x,\ 4.82x,\ 1.27x+5.09,\ 138.91-9.02x.$

Exercise V

Solve the following, doing all you can mentally:

1. $3x+5-x=15$
2. $5x-6+x=24$
3. $8-x+3x=32$
4. $8t-10-2t=50$
5. $15-6t+8t=25$
6. $6t-7+4t=13$
7. $7y-3y+10y=39$
8. $2(x-3)+12=18$
9. $8+5(s+7)=63$
10. $18r+13-10r=75$
11. $\frac{x}{4}+5=9$
12. $\frac{r}{3}+8-\frac{r}{5}=10$
13. $\frac{9x}{2}-\frac{5x}{3}=28$
14. $\frac{26y}{3}-15+\frac{5y}{2}=52$
15. $3.7l-3.6-2.9l=2.4$
16. $2x+4(x+10)+3x=130$
17. $\frac{3x}{4}-\frac{5x}{12}+\frac{x}{3}=8$
18. $7ax-ax+a=7a.$

Adjacent Angles

29. The point at which the sides of an angle meet is called the **vertex** of the angle.

Vertex is a Latin word meaning "turning-point."

1. On tracing paper make a trace of $\angle x$ (Fig. 43), and fit this trace on $\angle y$. How do the angles compare in size? Does the size of an angle depend on the lengths of its sides? Measure $\angle x$ and $\angle y$ with a protractor.

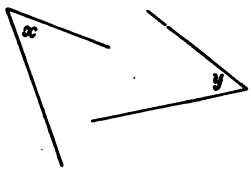


FIG. 43

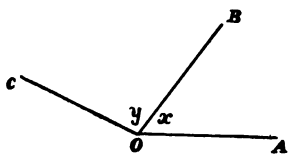


FIG. 44

2. Read the vertex and the sides of $\angle$A O B (Fig. 44); of $\angle$B O C.

3. Do the angles of Fig. 44 have a side in common? A common vertex?

30. Two angles that have the same vertex and a common side between them are **adjacent angles.**

The sides which are not common are called the **exterior sides.**

1. Are $\angle b$ and $\angle c$ (Fig. 45) adjacent? $\angle c$ and $\angle d$? $\angle b$ and $\angle a$? $\angle b$ and $\angle d$?

2. Read the exterior sides of $\angle x$ and $\angle y$ (Fig. 44); the common side.

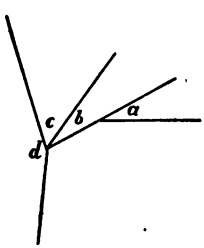


FIG. 45

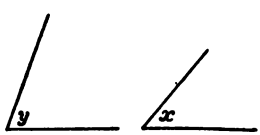


FIG. 46

31. The angles x and y (Fig. 46) may be added without measuring them, by placing them adjacent.

1. Express by an equation that $\angle$A O C (Fig. 47) equals the sum of two angles.

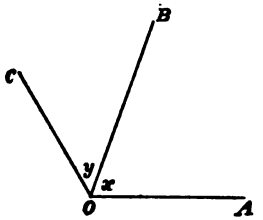


FIG. 47

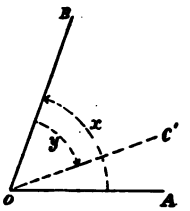


FIG. 48

2. If the angle y (Fig. 47) is turned over O B as a hinge, so as to make O C lie along the dotted line O C′ (Fig. 48), express the value of angle A O C′ in terms of $\angle x$ and $\angle y$.

3. Draw two angles, a and b, and show how to find the sum without measuring them. Show how to find the difference without measuring.

4. Show the sum and the difference of angles by folding or cutting paper.

32. With a protractor draw an angle of 60°.

Examine Figures 49 and 50 and construct on a line, A B, an angle of 60°, like B O C, Fig. 51.

FIG. 49

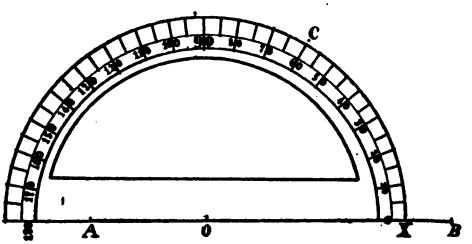


FIG. 50

1. With a protractor draw the following angles, marking on each angle the number of degrees, as in Fig. 51: 80°; 92½°; 170°.

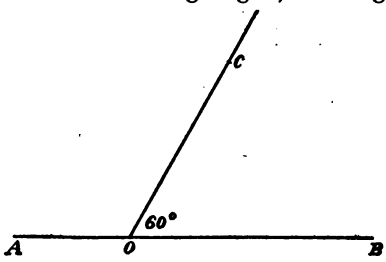


FIG. 51

2. With a protractor draw adjacent angles of 75° and 85°; of 103½° and 57½°; of 31½° and 2½°. Check the work, first, by finding the sum arithmetically, then, by measuring the *sum* with a protractor.

3. Draw adjacent angles of 56° and 124°; of 19½° and 160½°; of 92° and 88°. With a ruler, or straight edge, see if the exterior sides of each pair of angles form a straight line. What term is applied to each angle-sum?

4. Draw two straight lines cutting each other so as to make a pair of adjacent angles equal (with the aid of a protractor), and show that the angles are right angles.

33. If two straight lines intersect (cut each other) making a pair of *adjacent angles equal*, the angles are **right angles**, and either line is **perpendicular** to the other.

34. An angle *less* than a right angle is called an **acute** (sharp) angle. An angle *greater* than a right angle and *less* than two right angles is called an **obtuse** (blunt) angle.

1. Draw two adjacent acute angles whose sum is a right angle. Point out two perpendicular lines.

2. Draw two adjacent obtuse angles whose sum is 3 right angles.

3. Draw two adjacent angles, one obtuse and the other acute, whose sum is two right angles.

4. Draw a triangle of which all angles are acute. From the vertex of each angle draw a line at right angles to the opposite side.

5. Draw a triangle having an obtuse angle, and draw perpendiculars as in problem 4.

6. Draw a triangle having a right angle, and draw perpendiculars as in problem 4.

7. How many altitudes does a triangle have? How many bases?

Supplementary Angles

35. Two angles whose sum is a straight angle (180°) are **supplementary angles.** Either angle is said to be the **supplement** of the other.

If the supplementary angles are adjacent they are called **supplementary adjacent** angles.

1. In Fig. 51 read the angle which is the supplement of ∠B O C; of ∠C O A.

2. Draw two supplementary adjacent angles.

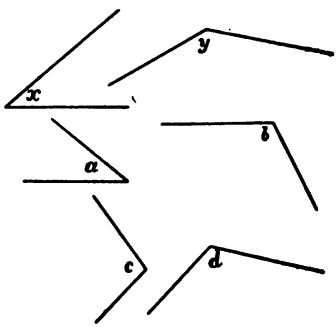


FIG. 52

3. On tracing paper make a trace of ∠x and of ∠y (Fig. 52) so that the angles on the tracing paper are adjacent. Show with a ruler, or straight edge, whether the angles are supplementary. In the same way show whether ∠a and ∠b are supplementary; ∠c and ∠d.

4. Are 50° and 130° supplementary? 37° and 133°? 60° and 120°? 90° and 90°?

5. How many degrees are there in the supplement of an angle of $45°$? Of $120\frac{1}{2}°$? Of $90°$? Of $a°$? Of $180°-x°$?

6. Write the supplement of an angle of x degrees.

7. Write the supplement of $a°$; of $b°$; of $5d°$; of $\frac{3c°}{2}$; of $y°+z°$.

8. If angles of $120°$ and $a°$ are supplementary, what does a represent?

9. If $x°+80°=180°$, what is the supplement of $x°$? What is the value of $x°$?

10. In the equation, $a°+b°=180°$, what is the supplement of $a°$? Of $b°$? Why?

11. State by an equation that the following pairs of angles are supplementary:

(1) $x°$ and $60°$	(4) $50°$ and $x°+70°$
(2) $70°$ and $y°$	(5) $2x°+3°$ and $27x°-2°$
(3) $b°$ and $c°$	(6) $\frac{3}{4}x°$ and $\frac{17}{5}x°+112\frac{1}{8}°$.

12. $x+3$ degrees is the supplement of $2x+27$ degrees. Find x, $x+3$, and $2x+27$.

We may write	$x+3+2x+27=180.$	Why?	(1)
Combining like terms	$3x+30=180.$	Why?	(2)
Subtracting 30	$3x=150.$	What axiom is used?	(3)
Dividing by 3	$x=\ 50.$	What axiom?	(4)
Whence	$x+3=\ 53.$	Why?	(5)
	$2x+27=127.$		(6)

CHECK: $53+127=180$.

13. $x°$ is the supplement of $x°+84°$. Find the angles.

14. One of two supplementary angles is $98°$ larger than the other. Find the angles.

15. One of two supplementary angles is $27°$ smaller than the other. Find the angles.

16. The difference of two supplementary angles is 110°. Find them. [Suggestion: Let x° be one angle and $x^\circ + 110^\circ$ the other.]

17. Find two supplementary angles whose difference is 21°; $36\frac{1}{2}^\circ$; $73\frac{1}{4}^\circ$; d°.

18. The difference between an angle and its supplement is 37°. Find the angle.

19. How many degrees are there in the angle, x°, if it is the supplement of $5x^\circ$? Of $7x^\circ$? Of $3\frac{1}{2}x^\circ$?

20. How many degrees are there in an angle that is the supplement of 4 times itself? Of 8 times itself? Of 10 times itself? Of $2\frac{1}{2}$ times itself? Of $\frac{3}{4}$ of itself? Of $\frac{1}{8}$ of itself?

21. Write in algebraic language—

the double of an angle, x
15° added to 3 times the angle
29° subtracted from 6 times the angle
4 times the sum of the angle and 13°
two-thirds of the sum of the angle and 17°.

22. Write in algebraic symbols—

(1) the supplement of an angle, x
(2) 5 times the supplement
(3) 3 times the supplement
(4) 14° added to 3 times the supplement
(5) 16° subtracted from 3 times the supplement
(6) the supplement increased by 10°
(7) the supplement diminished by 18°
(8) the supplement divided by 4
(9) one-third of the supplement
(10) 17° added to the supplement
(11) 20° added to one-third of the supplement
(12) 19° subtracted from $\frac{3}{5}$ of the supplement.

23. If an angle is doubled and its supplement is increased by 20°, the sum of the angles obtained is 280°. Find the two supplementary angles. [Suggestion: Let x be one angle, and $180-x$ the other; then by the conditions of the problem $2x+180-x+20=280$.]

24. If an angle is trebled, and its supplement is diminished by 112°, the sum of the angles obtained is 168°. Find the supplementary angles.

25. The sum of an angle and $\frac{1}{3}$ of its supplement is 90°. Find the angle.

26. If an angle is increased by 12°, and its supplement is divided by 5, the sum of the angles obtained is 80°. Find the supplementary angles.

27. If 20° is added to 5 times an angle, and 15° is subtracted from 2 times the supplement of the angle, the sum of the angles obtained is 401°. Find the supplementary angles.

Find x and each angle of the following pairs of supplementary angles. With protractor draw figures for 28, 29, and 30:

28. $\frac{3}{5}x$, $88+\frac{1}{5}x$

29. $\frac{2}{3}x+14$, $82-\frac{1}{3}x$

30. $\frac{5}{7}x+29$, $97-\frac{2}{7}x$

31. $\frac{x}{3}+161$, $\frac{x}{2}-26$

32. $\frac{x}{3}+97$, $\frac{x}{4}+69$

33. $\frac{3x}{4}+37$, $109+\frac{2x}{3}$

34. $87+\frac{x}{5}$, $\frac{x}{3}+69$

35. $77+\frac{2x}{3}$, $59+\frac{4x}{5}$

36. $\frac{2x}{3}+93$, $\frac{x}{2}-18$

37. $23+\frac{3x}{4}$, $136-\frac{2x}{5}$

38. $3(x+8)$, $\frac{1}{2}(4x+88)$

39. $\frac{1}{3}(2x+15)$, $78+\frac{3}{4}x$

40. $\frac{1}{2}(x-36)$, $\frac{1}{3}(279+2x)$

41. $\frac{1}{4}(3x+148)$, $\frac{1}{3}(2x+327)$.

Exercise VI

Solve the following:

1. $\frac{3x}{5}+88+\frac{x}{5}=180$
2. $\frac{4t}{7}+10-\frac{3t}{14}=15$
3. $\frac{z}{3}+16+\frac{z}{2}-14=7$
4. $\frac{2r}{5}-15+\frac{3r}{4}=8$
5. $9+\frac{x}{5}-12+x=9$
6. $2(x-5)+\frac{1}{2}(x+1)=3$
7. $5(x-5)+(x+3)=2$
8. $\frac{15z}{2}-20+(z+12)=9$
9. $\frac{x+8}{3}+8=12$
10. $\frac{3x-2}{7}-1=\frac{8}{7}$
11. $\frac{5(x+4)}{11}+8=13$
12. $9x-\frac{x}{9}-1=7\frac{8}{9}$
13. $\frac{15r}{7}+15-\frac{3r}{14}=28\frac{1}{2}$
14. $\frac{x-\frac{3}{4}}{2}+10+x=10\frac{3}{4}$
15. $\frac{x+5}{3}+\frac{2x-5}{5}-4=4$
16. $\frac{6(x-2)}{3}+\frac{x+3}{4}-3=5$
17. $\frac{7(y+5)}{3}-\frac{3y-1}{2}-9=4.$

Vertical Angles

36. Two angles having a *common vertex*, and having sides in the *same straight line*, but in *opposite directions*, are called **opposite** or **vertical** angles (as x and z, Fig. 53).

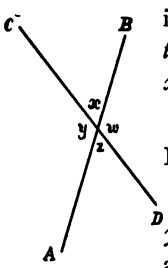


Fig. 53

1. Read both pairs of vertical angles in Fig. 53.

2. On tracing paper make a trace of angles y and z (Fig. 53). Put this trace on angles x and w and see whether $\angle z$ coincides with (fits on) $\angle x$, and $\angle y$ with $\angle w$. How do the vertical angles compare in size?

3. Test your conclusion in problem 2 by drawing two intersecting straight lines and measuring both pairs of vertical angles with a protractor.

4. Are $\angle$s x and y (Fig. 53) adjacent? Are they supplementary? What is their sum?

5. Are $\angle$s y and z (Fig. 53) supplementary adjacent? What is their sum?

6. From problems 4 and 5 show that

$$x+y=y+z.$$

7. Show how to get the equation

$$x=z$$

from the equation in problem 6. What axiom must be used?

8. Using the equation

$$y+x=x+w,$$

show as in problems 4, 5, 6, and 7 that

$$y=w.$$

37. The problems of section **36** show the truth of the following theorem or proposition:

THEOREM. *If two lines intersect, the vertical angles are equal.*

1. Show that the above theorem is true for *any* two intersecting straight lines, no matter what angles they make.

2. State the three ways in which the truth of the foregoing theorem was shown in **§36.** By which of these ways was the truth of the theorem shown for *any* pair of vertical angles?

3. Find the values of the four angles of Fig. 54.

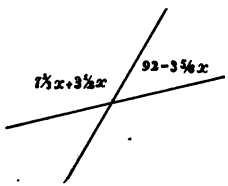


FIG. 54

We may write

$$7\tfrac{2}{3}x+3\tfrac{1}{2}x+92-3\tfrac{5}{6}x=180. \quad \text{Why?} \qquad (1)$$

Equation (1) may be written

$$7\tfrac{2}{3}x+3\tfrac{1}{2}x-3\tfrac{5}{6}x+92=180. \quad \text{Why?} \qquad (2)$$

$$7\tfrac{1}{3}x+92=180. \quad \text{Why?} \qquad (3)$$

Subtracting 92 $\quad 7\tfrac{1}{3}x=88.$ What axiom? (4)

Multiplying by 3 $\quad 22x=264.$ What axiom? (5)

Dividing by 22 $\quad x=12.$ What axiom? (6)

$$7\tfrac{2}{3}x+3\tfrac{1}{2}x=7\tfrac{2}{3}\cdot 12+3\tfrac{1}{2}\cdot 12=92+42=134.$$
$$92-3\tfrac{5}{6}x=\quad 92-3\tfrac{5}{6}\cdot 12=92-46=46.$$

The other two angles are also 134° and 46°. Why?

CHECK: $134+46+134+46=360$.

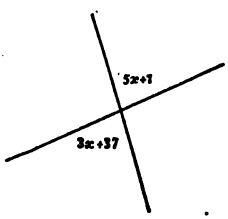


FIG. 55

4. Find the four angles made by two intersecting straight lines, if two adjacent angles are $9x+41$ and $5x-29$. Check results by drawing a figure.

5. Find x and the four angles made by two intersecting straight lines, if two vertical angles (Fig. 55) are $3x+37$ and $5x+7$.

Since the given angles are vertical angles,

$$5x+7=3x+37. \qquad (1)$$

Subtracting 7 $\quad 5x=3x+30.$ What axiom? (2)

Subtracting $3x$ $\quad 2x=30.$ What axiom? (3)

Dividing by 2 $\quad x=15.$ What axiom? (4)

$$5x+7=5\cdot 15+7=82.$$
$$3x+37=3\cdot 15+37=82.$$

Each of the other two angles is $180-82=98$. Why?

CHECK: $82+98+82+98=360$.

Find x and each of the following angles made by two intersecting straight lines. Draw figures for 6, 7, 16, and 17:

VERTICAL PAIRS

6. $7x+27$ and $4x+87$
7. $3x-17$ and $x+103$
8. $\frac{3}{5}x+16\frac{1}{2}$ and $\frac{2}{5}x+24\frac{1}{2}$
9. $2\frac{4}{11}x-13$ and $1\frac{1}{11}x+57$
10. $\frac{1}{8}x+\frac{5}{8}x$ and $\frac{8}{2}x+42$
11. $\frac{3}{8}x+\frac{1}{2}x-28$ and x
12. $5x+\frac{3x}{4}$ and $\frac{5x}{2}+130$
13. $\frac{7x}{4}-\frac{x}{6}$ and $\frac{2x}{3}+66$
14. $\frac{x}{3}+\frac{x}{6}$ and $\frac{x}{4}+18$
15. $\frac{7x}{2}-\frac{3x}{4}$ and $\frac{3x}{5}+8\frac{3}{5}$.

ADJACENT PAIRS

16. $\frac{x}{5}+\frac{x}{2}+172\frac{1}{2}$ and $\frac{3x}{10}-\frac{x}{4}$
17. $x-\frac{x}{7}$ and $\frac{3x}{4}+90$
18. $\frac{2x}{5}+\frac{x}{3}$ and $154\frac{1}{2}-\frac{x}{6}$
19. $\frac{x}{4}+4x$ and $87-\frac{5x}{3}$
20. $6\frac{5}{6}x-2\frac{1}{2}x$ and $4\frac{3}{4}x-365$
21. $\frac{x}{4}+2x$ and $\frac{x}{6}+35$
22. $\frac{1}{11}(26x-143)$ and $\frac{1}{11}(12x+33)$
23. $2\left(\frac{x}{5}+\frac{x}{6}\right)$ and $\frac{1}{2}\left(309-\frac{x}{3}\right)$
24. $2\left(\frac{x}{2}+x\right)$ and $\frac{1}{6}(x+210)$
25. $\frac{1}{3}(261-5x)$ and $4\left(x+\frac{x}{16}\right)$.

Exercise VII

Solve the following, doing all you can mentally:

1. $2x+3=x+5$
2. $5x-6=2x$
3. $4t+3=t+12$
4. $3s-2=2s+7$
5. $7y-7=3y+21$
6. $5(x-3)=3x+3$
7. $2(2x+1)=3x+5$
8. $6(z-7)=z+8$
9. $7(3z-2)=5z+2$
10. $10(r-\frac{1}{2})=10r-5$

*11. $3(x+1)=8(x-4)$

12. $\frac{3x}{5}+4=x+2$
13. $\frac{4x}{7}+10=x+4$
14. $8z-\frac{z}{3}=2z+17$
15. $\frac{5z}{8}-\frac{z}{16}=\frac{z}{2}+\frac{1}{2}$
16. $\frac{x+1}{3}=x-7$
17. $4(x-7)=3x+14$
18. $\frac{x}{3}-\frac{x}{5}=17-x$
19. $\frac{7(s+3)}{12}+s=\frac{s}{3}+28$
20. $ax+a=9a$
21. $bz-3b=5b$
22. $\frac{by}{3}+b=\frac{7b}{5}$.

*Suggestion.—After multiplying as indicated, subtract $3x$ from both sides of the equation, and add 32.

It is sometimes convenient, in solving equations, to add or to subtract such a number or such numbers that the term containing the *number whose value is required* may stand alone on the *right side.*

Complementary Angles

38. Two angles whose sum is a right angle are **complementary angles.**

Either angle is said to be the **complement** of the other.

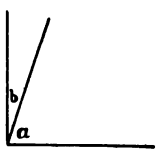


Fig. 56

1. What is the complement of $\angle a$ (Fig. 56)? Of $\angle b$?

2. Draw two adjacent complementary angles. May either angle be obtuse? Point out two perpendicular lines.

3. On tracing paper make a trace of $\angle x$ and of $\angle y$ (Fig. 57) so that the angles on the tracing paper are adjacent. With a protractor show whether the angles are complementary. In the same way test whether $\angle a$ and $\angle b$ are complementary; $\angle c$ and $\angle d$.

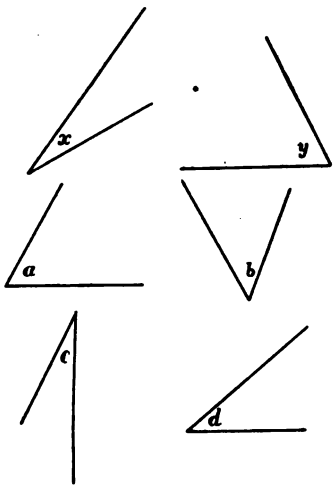


FIG. 57

4. Show whether $22°$ and $68°$ are complementary; $43\frac{2}{3}°$ and $46\frac{2}{3}°$; $89\frac{3}{8}°$ and $\frac{5}{8}°$.

5. What is the complement of $60°$? Of $30°$? Of $10\frac{3}{8}°$? Of $45\frac{5}{7}°$? Of $a°$? Of $90°-x°$?

6. Write the complement of n degrees.

7. Write the complement of $d°$; of $3c°$; of $\frac{5d°}{3}$; of $x°+y°$; of $\frac{3s°+2t°}{5}$; of $7(a+b)$ degrees; of $5x^2$ degrees; of $7y^3$ degrees; of $3x^2-5y^4$ degrees.

8. If angles of $40°$ and $d°$ are complementary, how many degrees does d stand for?

9. If $y°+70°=90°$, what is the complement of $y°$? Why? What is the value of $y°$?

10. In the equation, $c°+d°=90°$, what is the complement of $c°$? Of $d°$? Why?

11. State by equations that the following pairs of angles are complementary:

(1) y° and 50° (2) 30° and z° (3) w° and x°
(4) $a^\circ+30^\circ$ and $a^\circ-20^\circ$ (5) $2x^\circ+7^\circ$ and $5x^\circ-2^\circ$
(6) $3(x+7)$ degrees and $5(2x-8)$ degrees
(7) $\frac{2}{3}x-15\frac{7}{9}$ degrees and $26\frac{1}{2}x+43\frac{5}{8}$ degrees.

12. If $\frac{n}{2}+55^\circ$ is the complement of $\frac{n}{3}-\frac{n}{4}$, find n, $\frac{n}{2}+55^\circ$, and $\frac{n}{3}-\frac{n}{4}$.

We may write

$$\frac{n}{2}+55+\frac{n}{3}-\frac{n}{4}=90. \quad \text{Why?} \qquad (1)$$

Rearranging the terms of equation (1)

$$\frac{n}{2}+\frac{n}{3}-\frac{n}{4}+55=90. \quad \text{Why?} \qquad (2)$$

Combining the n-terms

$$\frac{7n}{12}+55=90. \qquad (3)$$

Subtracting 55 $\quad \frac{7n}{12}=35. \quad$ Why? (4)

Multiplying by 12 $\quad 7n=420. \quad$ Why? (5)

Dividing by 7 $\quad n=60. \quad$ Why? (6)

$$\frac{n}{2}+55=\frac{60}{2}+55=85.$$

$$\frac{n}{3}-\frac{n}{4}=\frac{60}{3}-\frac{60}{4}=5.$$

CHECK: $85+5=90$.

13. x° is the complement of $x^\circ+48^\circ$. Find the angles.

14. One of two complementary angles is 24° larger than the other. Find the angles.

15. One of two complementary angles is 28° smaller than the other. Find the angles.

16. The difference of two complementary angles is 83°. Find them.

17. Find two complementary angles whose difference is 21°; $36\frac{1}{2}^\circ$; $73\frac{1}{4}^\circ$; d°.

18. The difference between an angle and its complement is $27°$. Find the angle.

19. How many degrees are there in the angle, x, which is the complement of $4x$? Of $6x$? Of $5\frac{1}{2}x$?

20. How many degrees are there in an angle that is the complement of 3 times itself? Of 7 times itself? Of 6 times itself? Of $3\frac{1}{3}$ times itself? Of $\frac{4}{5}$ of itself? Of $\frac{1}{5}$ of itself?

21. Write in symbols—

the double of the angle y
$17°$ added to 3 times the angle
$31°$ subtracted from 5 times the angle
6 times the sum of the angle and $19°$
$\frac{3}{4}$ of the sum of the angle and $22°$
5 times the angle minus $16°$.

22. Write in symbols—

(1) the complement of an angle y
(2) 6 times the complement
(3) 4 times the complement
(4) $19°$ added to 4 times the complement
(5) $17°$ subtracted from 4 times the complement
(6) the complement of y increased by $12°$
(7) the complement diminished by $25°$
(8) the complement divided by 7
(9) one-fifth of the complement
(10) $12°$ added to the complement
(11) $13°$ added to one-fourth of the complement
(12) $18°$ subtracted from $\frac{3}{7}$ of the complement.

23. If an angle is doubled, and its complement is increased by $40°$, the sum of the angles obtained is $160°$. Find the complementary angles.

24. If an angle is trebled, and its complement is diminished by $40°$, the sum of the angles obtained is $130°$. Find the complementary angles.

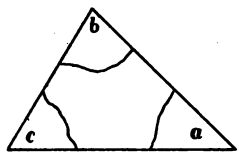
25. The sum of an angle and $\frac{1}{2}$ of the complement is $75°$. Find the angle.

26. If an angle is increased by $15°$, and the complement is divided by 3, the sum of the angles obtained is $75°$. Find the complementary angles.

27. If $20°$ is added to 3 times an angle, y, and $6°$ is subtracted from $\frac{2}{3}$ of the complement of y, the sum of the angles obtained is $102°$. Find the complementary angles.

The Sum of the Angles of a Triangle

1. Measure the angles of a triangle with a protractor and find the sum.



2. Draw and cut out a triangle. Tear off the corners, and place them as in Fig. 58. What seems to be the sum of the three angles of the triangle? Test with a ruler, or straight edge.

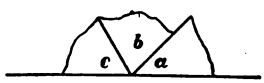


FIG. 58

3. Draw two triangles different in size and shape from the triangle used in problem 2, and find the sum of the angles as in problem 2.

4. Draw a triangle. From a vertex draw a line (B O, Fig. 59) at right angles to the opposite side. Cut out the triangle and fold so that the vertices come together at the foot of the perpendicular (at O, Fig. 59). What seems to be the sum of the three angles of the triangle?

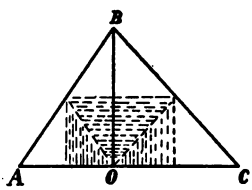


FIG. 59

5. Draw two triangles different in size and shape from the triangle

used in problem 4, and find the sum of the angles as in problem 4.

6. Draw a triangle. Place a pencil or stick in the position 1 (Fig. 60), noting the direction it is pointing. Rotate the pencil through the ∠s x, y, and z, successively, as indicated in the figure.

Through what part of a complete turn has the pencil rotated? Through how many right angles? How many degrees?

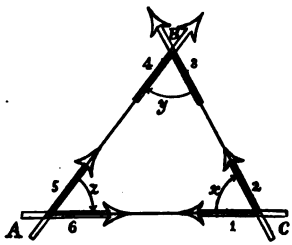


FIG. 60

7. State by an equation the number of degrees in the sum of ∠s x, y, and z (Fig. 60).

39. If one side of the triangle A B C (Fig. 61) is prolonged at each vertex, the angles w, s, and t are called the **exterior** (outside) angles of the triangle A B C.

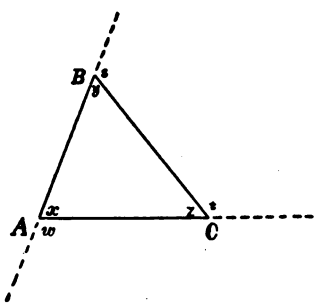


FIG. 61

Angles x, y, and z are called the **interior** (inside) angles of the triangle A B C.

40. If one angle of a triangle is a right angle, the triangle is called a **right triangle.**

1. State by an equation the sum of the interior angles of triangle A B C (Fig. 61).

2. How many degrees in $x+w$ (Fig. 61)? In $y+s$? In $z+t$?

3. Show that

$$x+w+y+s+z+t=180+180+180=540.$$

4. From the equation in problem 3 show that

$$w+s+t=360.$$

The equation in problem 3 may be written

$$x+y+z+w+s+t=540. \quad \text{Why?} \qquad (1)$$

But

$$x+y+z=180. \quad \text{Why?} \qquad (2)$$

If the left side of equation (2) is subtracted from the left side of equation (1), and the right side of equation (2) from the right side of equation (1), the result is

$$w+s+t=360. \quad \text{Why?} \qquad (3)$$

5. Translate equation (3) of problem 4 into words.

41. We have considered the following theorems (propositions) about the angles of triangles:

THEOREM I. *The sum of the interior angles of a triangle is 180°.*

THEOREM II. *The sum of the exterior angles of a triangle, taking one at each vertex, is 360°.*

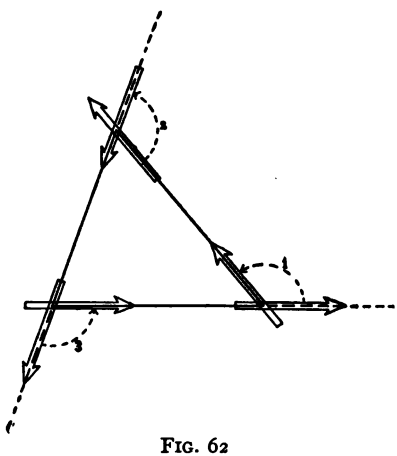


FIG. 62

1. Prove Theorem II, §41, by rotating a pencil as indicated in Fig. 62.

2. The interior angles of a triangle are $3x$, x, $6x$. Find their values.

3. Find the value of each angle of a triangle, in degrees, if the first angle is twice the second, and the third is three times the first.

4. Find the angles of a triangle if the first angle is 6 times the second, and the third is $\frac{1}{2}$ of the first.

5. Solve a problem like 4, supposing the third angle to be $\frac{1}{3}$ of the first; $\frac{2}{3}$ of the first; $\frac{1}{5}$ of the first; $\frac{5}{7}$ of the first.

6. Find the angles of a triangle if the first is $\frac{1}{3}$ of the second, and the third is $\frac{1}{2}$ of the first.

7. Find the angles of a triangle if the first angle is 18° more than the second, and the third is 12° less than the second.

8. The difference between two angles of a triangle is 20°, and the third angle is 36°. Find the unknown angles.

9. Find the angles of a triangle if the first is 25° more than the second, and the third is 3 times the first.

10. Find the angles of a triangle if the first angle is double the second, and the third is 3 times the first, less 9°.

11. Find the angles of a triangle if the first is $\frac{1}{4}$ of the second, and the third is $\frac{1}{7}$ of the first, plus 18°.

12. Find the angles of a triangle if the first is $3\frac{1}{2}$ times the second, minus 8°, and the third is $\frac{1}{5}$ of the second.

13. Find the angles of a triangle if the first is 6 times the second, plus 18°, and the third is $\frac{1}{2}$ of the first, minus 7°.

14. If one angle of a triangle is a right angle, what is the sum of the other two angles? What are such angles called?

15. How many angles of a triangle may be right angles? Obtuse angles? Acute angles?

16. Find the values of the acute angles of a right triangle if one angle is

(1) 3 times the other (2) 5 times the other
(3) $\frac{2}{3}$ of the other (4) $1\frac{1}{5}$ of the other
(5) 6 more than 7 times the other
(6) $\frac{1}{2}$ of the other, diminished by 33.

17. The acute angles of a right triangle are $\frac{n}{2}$ and $\frac{n}{3}$. Find the values of n and of the acute angles.

18. The acute angles of a right triangle are equal. Find them.

19. Find the unknown interior angles of the triangles of figures 63, 64, 65, 66. (Suggestion: Use the theorems on vertical angles, and on the angle-sum of a triangle.)

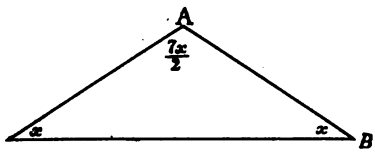


FIG. 63

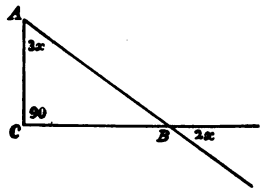


FIG. 64

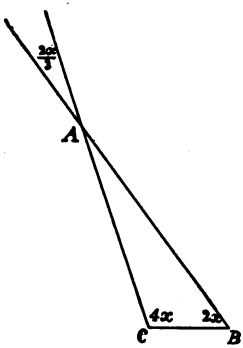


FIG. 65

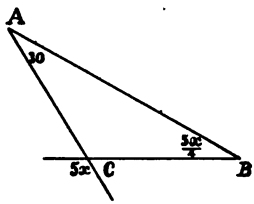


FIG. 66

20. The three interior angles of a triangle are equal. Find them.

21. The three exterior angles of a triangle are equal. Find the value of each exterior and interior angle.

EXERCISE VIII

Solve the following:

1. $6x-2=26$
2. $9x-8=x$
3. $5z+6=11$
4. $2(z-3)=4$
5. $3(y+2)=5y$
6. $2y-7=y+3$

7. $6r-5=4r+7$

8. $\frac{r}{2}+\frac{r}{3}=5$

9. $\frac{4s}{5}-\frac{s}{3}=7$

10. $\frac{s}{7}-\frac{s}{8}=1$

11. $\frac{7t}{6}-5\frac{1}{2}=8-\frac{13t}{12}$

12. $\frac{7}{8}t=\frac{7}{8}(3-2t)$

13. $4x-19=\frac{2x+21}{4}+\frac{1}{4}$

14. $2x-1=\frac{7(5-2x)}{3}+\frac{2}{3}$

15. $6+3y=\frac{3(5+2y)}{4}+9\frac{3}{4}$

16. $7z-8=6z+\frac{z}{5}$

17. $5a+1=\frac{8-3a}{5}+5$

18. $\frac{6a-1}{5}+a=4-2a$

19. $2l+2=\frac{7l+5}{6}+4\frac{1}{2}$

20. $l+\frac{3l}{2}=\frac{12-7l}{2}$

21. $6x-a-2x=3a$

22. $8(x-a)=8+4a$

23. $x-\frac{a}{2}+2x+\frac{3a}{2}=4a$

24. $x-2a+\frac{3}{5}=1-\frac{2}{5}$

25. $\frac{x}{a}-\frac{x}{2a}=3$

26. $\frac{x}{b}+\frac{3x}{2b}=6$

27. $z+1=\frac{7z}{16}-\frac{z}{8}+12$

28. $\frac{2+5z}{5}+5z=2.$

Summary

1. An **angle** is formed by a line rotating in a plane about a point called the **vertex.**

An angle may also be formed by two intersecting lines.

2. A **straight angle** is the angle made by a *half-turn* of a rotating line.

3. A **right angle** is *half of a straight angle,* or the angle made by a *quarter-turn* of a rotating line.

4. An **acute angle** is *less* than a right angle. An **obtuse angle** is *greater* than a right angle and *less* than a straight angle.

5. A **degree** is one-nintieth of a right angle.

6. Two angles are **adjacent** if they have the *same vertex*, and a *common side between them.*

7. If two straight lines intersect so as to make the *adjacent angles equal*, the angles are **right angles,** and either line is **perpendicular** to the other.

8. Angles may be added and subtracted without measuring.

9. The sum of the angles having a common vertex, and *just covering the plane* about the vertex, is 360°, or four right angles.

10. The sum of the angles having a common vertex, and just covering the part of the plane *on one side of a straight line* is 180°, or two right angles.

11. Two angles are **supplementary** when their sum is two right angles, or 180°. If the supplementary angles are *adjacent* they are called **supplementary adjacent angles.**

12. The *exterior sides* of supplementary adjacent angles lie in the *same straight line.*

13. **Opposite** or **vertical** angles are angles that have a *common vertex*, and have sides in the *same straight lines*, but in *opposite directions.*

14. Two angles are **complementary** when their sum is a right angle, or 90°.

15. The truth of the following theorems has been shown:

THEOREM I. *If two straight lines intersect, the opposite or vertical angles are equal.*

THEOREM II. *The sum of the interior angles of a triangle is 180°, or two right angles.*

THEOREM III. *The sum of the exterior angles of a triangle, taking one at each vertex, is 360°, or four right angles.*

16. A triangle having a *right angle* is a **right triangle.**

17. *At least two* angles of a triangle must be *acute.*

18. The *acute* angles of a *right* triangle are *complementary.*

19. The equation may be used to express the relations between angles, and to solve problems on angles.

20. The *order of terms* of a polynomial may be changed, without changing the *value* of the polynomial, provided each term as it changes position carries with it the sign next before it.

CHAPTER IV

POSITIVE AND NEGATIVE NUMBERS

Uses of Positive and Negative Numbers

42. The plus (+) and minus (−) signs are used to denote numbers supposed to be used in certain directions, called *directed* numbers.

1. The top of the mercury column of a thermometer stands at 0° at the beginning of an hour. The next hour it rises 5° and the next 3°. What does the thermometer read?

2. If the mercury stands at 0°, and rises 8°, then falls 5°, what does the thermometer read?

3. Denoting a *rise* of 10°, or of x° on the thermometer by R 10°, or by R x°, and a fall of 10°, or of x°, by F 10°, or F x°, give the readings of the thermometer after the following changes, if the top of the column reads 0° at the start:

(1) R 8°	followed by R 5°	(4) R 13°	followed by F 18°
(2) R 12°	" by F 9°	(5) F 5°	" by R x°
(3) R 16°	" by F 12°	(6) R a°	" by F b°.

4. If the change in the mercury column is a rise, a *positive* or *plus* (+) sign will be written before the number that denotes the *amount* of the change. If the change is a *fall*, a *negative* or *minus* (−) sign will be written before the number. If the reading at the start is 0°, give the readings after these changes:

(1) +10	followed by + 2	(5) $+x$	followed by $+y$
(2) +10	" by − 2	(6) $+a$	" by $-x$
(3) +20	" by −18	(7) $+a$	" by $-a$
(4) + 9	" by −12	(8) $-a$	" by $-x$.

5. On a winter day the thermometer was read at 9 A. M. and every hour afterward until 5 o'clock. The hourly readings were $-5°$, $0°$, $+2°$, $+8°$, $+10°$, $+10°$, $+5°$, $0°$, $-5°$.

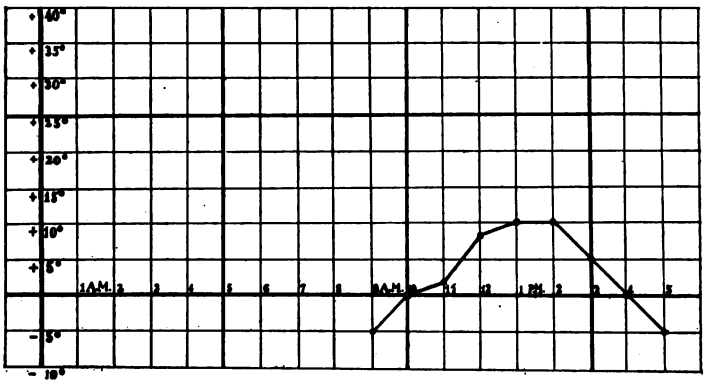


FIG. 67

On squared paper the readings were marked off from hour to hour, calling one vertical space $5°$. The points were connected as shown. How did the mercury change from 9 to 10 o'clock? From 10 to 11 o'clock? From 11 to 12? From 12 to 1? From 1 to 2? From 2 to 3? From 3 to 4? From 4 to 5? Tell for each hour whether the change was a rise, or a fall.

6. Draw a line to show the following hourly readings, beginning at 8 A. M. Indicate the hourly changes in amount by figures, and in direction by the + and − signs:

$+2°$, $-2°$, $-4°$, $-2°$, $+2°$, $+4°$, $+4°$, $+4°$, $+8°$, $+10°$.

7. The average monthly temperatures for a northern town are

Jan.	$-4°$	May	$+42°$	Sept.	$+48°$
Feb.	$-7°$	June	$+52°$	Oct.	$+37°$
Mar.	$+14°$	July	$+62°$	Nov.	$+25°$
April	$+26°$	Aug.	$+60°$	Dec.	$+2°$.

Using a convenient scale, draw the temperature-line.

43. Drawing temperature-lines is called **graphing,** or **plotting.**

1. The daily average temperatures for 14 days at a certain place were $+8°$, $0°$, $-10°$, $+12°$, $-6°$, $+14°$, $+15°$, $+2°$, $-5°$, $+15°$, $+20°$, $0°$, $0°$, $+10°$. Graph these readings to a convenient scale.

2. A bicyclist starts from a point and rides 18 miles due northward ($+18$ mi.) then 10 mi. due southward (-10 mi.); how far is he then from the starting-point?

3. State how far and in what direction from the starting-point a bicyclist would be after rides indicated by each of these pairs of records:

(1) $+10$ mi. then $-$ 8 mi.	(3) $+100$ mi. then $+50$ mi.
(2) -20 mi. " $+20$ mi.	(4) $+$ a mi. " $+$ b mi.

4. How far and in what direction from the starting-point is a traveler who goes eastward ($+$) or westward ($-$) as shown by these pairs of numbers:

(1) $+16$ mi. then $-$ 6 mi.	(4) $+a$ mi. then $+c$ mi.
(2) -18 mi. " $+28$ mi.	(5) $+m$ mi. " $-n$ mi.
(3) $-m$ mi. " $+$ 3 mi.	(6) $-m$ mi. " $+n$ mi.

5. A car in the middle of a moving train is drawn forward with a force of 8 tons and at the same time it is pulled backward with a force of $7\frac{1}{2}$ tons. The two forces together are equal to what single force?

6. Denoting a *forward* pulling force by F and a *backward* by B, give amount and direction of a single force equal to each of these pairs of forces:

(1) F 14 oz. with B 6 oz.	(3) F 25 tons with B 15 tons
(2) F 20 lb. " B 12 lb.	(4) B 25 tons " F 40 tons.

7. Denoting forward-pulling forces by the positive or plus ($+$) sign and back-pulling forces by the negative or minus

(−) sign, give the single force which is equal to each of these pairs of forces:

(1) +20 and −12	(4) −15 and + 8	(7) $+a$ and $+b$
(2) +20 " −20	(5) −12 " −12	(8) $+a$ " $-b$
(3) −15 " − 8	(6) + x " −12	(9) $-a$ " $-b$.

8. A toy balloon pulls upward with a force of 9 oz. If a weight of 6 oz. is attached, will the balloon rise or fall? With what force?

9. Call upward forces positive, or plus (+), and downward forces negative, or minus (−). State what single force will have the same effect as these pairs:

(1) +17 lb. and − 7 lb.	(4) −23 lb. and +10 lb.
(2) +17 lb. " −10 lb.	(5) + x lb. " + y lb.
(3) −23 lb. " −10 lb.	(6) + x lb. " − x lb.

10. Denoting motion *northward* by the positive or plus (+) sign and motion *southward* by the negative or minus (−) sign, and supposing a ship to start on the equator and sail as indicated, tell the latitude of the ship in both amount and sign for each pair of sailings:

(1) +28° then + 2°	(5) $+x°$ then −10°
(2) + 2° " −18°	(6) $-x°$ " −10°
(3) +12° " −12°	(7) $+x°$ " − $y°$
(4) +12° " −24°	(8) $-x°$ " − $y°$.

11. A boy starts work with no money. He earns 50¢ (+50¢) and spends 40¢ (−40¢). How much money has he then?

12. If a man's debts be indicated by writing D before their amount and his possessions (assets) by P before their amount, what is the condition of a man's affairs if his debts and possessions are indicated by P $1,200 and D $1,000? by P $73 and D $50? D $75 and P $60? D $300 and P $1,000?

13. If water pushes (buoys) a floating body upward with a force of 18 lb., and the body's weight pulls it downward with a force of 10 lb., the two forces together equal what single force?

14. If a man was born 40 B. C. and died 45 A. D., how old was he when he died?

15. Denoting a date A. D. by + and B. C. by −, give the length of time between these pairs of dates:

(1) -5 to $+10$ (3) -52 to -50 (5) -150 to $+150$
(2) $+16$ to $+86$ (4) -100 to $+50$ (6) $+1600$ to $+1900$.

16. Virgil was born -70 and died -19; how old was he at death?

17. The first Punic War lasted from -264 to -241; how long did it last?

18. Egypt was a Roman Province from -30 to $+616$; how many years was this?

19. Augustus was Emperor of Rome from -50 to $+14$; how many years was he Emperor?

20. What will denote the distance and direction from your school house to your home, if the distance and direction from your home to your school house are denoted by $+60$ rd.? $+1\frac{1}{2}$ mi.? $+x$ rd.? -80 rd.? $-1\frac{1}{4}$ mi.? $-a$ mi.?

21. While a freight train is moving at the rate of 10 mi. an hour toward the south ($+10$ mi. an hr.) a brakeman walks along the top of the cars toward the north at the rate of 4 mi. an hour (-4 mi. an hr.). How fast and in what direction does the brakeman move over the ground? Answer with the aid of the plus, or minus, sign.

22. The conductor of a passenger train walks from the front toward the rear of the train at the rate of 3 mi. an hour while the train is running at the rate of 12 mi. an hour. How

fast does the conductor move over the ground? Answer with the aid of the (+), or (−), sign, supposing that + means toward the north and first, that the train is running north; then second, that the train is running south.

23. What does the sign (−) denote if the sign (+) denotes: (1) above? (2) forward? (3) upward? (4) to the right? (5) after? (6) east? (7) north? (8) possessions?

44. The foregoing problems show the need for distinguishing *numbers of opposite nature.* Positive and negative signs afford a convenient means of making this distinction.

45. A number that denotes both magnitude and direction, or both size and quality, is called a *directed number*, as 16 units east, or $+16$, 10 units west, or -10, ax units downward, or $-ax$. If the direction or quality is denoted by the sign (+) or (−), the number is called a *signed number.*

Either of the two opposite directions, or qualities, may be denoted by the plus (+) sign, whereupon the opposite direction, or quality, is denoted by the minus (−) sign.

46. The plus and minus signs are also used, as in arithmetic, to denote *addition* and *subtraction.*

47. It is sometimes desirable to indicate *definitely* that the sign is to denote direction, or quality, when used with a number. This is done by inclosing the number together with the sign of quality in a parenthesis, thus, $(+6)$, (-8), $(+3x)$, $(-2y)$.

For example, the expression, $6+(+2)$, means that $+2$ is to be added to 6; the expression, $7-(+4)$, means that $+4$ is to be subtracted from 7.

Give the meaning of the following:

1. $6-(+3)$
2. $8+(-4)$
3. $-5+(+7)$
4. $-6-(-9)$
5. $a+(-b)$
6. $m-(-x)$.

48. The positive sign (+) need not always be written. It is generally omitted from the first number in an expression. The negative sign is never omitted. An expression like $+x-a$, where the first number, x, is positive, would commonly be written $x-a$, and is read "x minus a." The positive sign, when omitted, is said to be "understood."

Graphing Data

1. Using a convenient scale, and calling the verticals age-lines, graph these average heights of boys and girls:

Age	Boys	Girls	Age	Boys	Girls
2 yr.	1.6 ft.	1.6 ft.	12 yr.	4.8 ft.	4.5 ft.
4	2.6	2.6	14	5.2	4.8
6	3.0	3.0	16	5.5	5.2
8	3.5	3.5	18	5.6	5.3
10	4.0	3.9	20	5.7	5.4.

At what age do boys grow most rapidly? Girls?

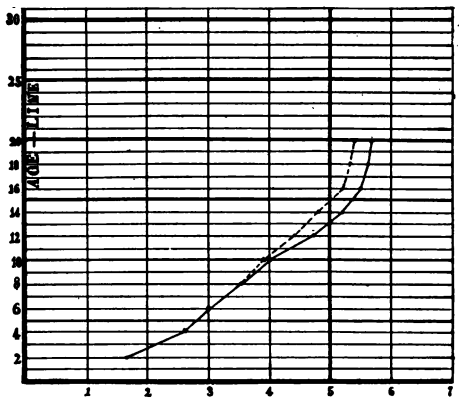


FIG. 68

2. The populations, in millions, of the United States for each 10 years beginning 1790, are 3.9, 5.3, 7.2, 12.9, 17.1, 23.2, 31.4, 38.6, 50.2, 62.6, 76.3.

Graph these numbers, drawing a smooth, free-hand curve through the points and predict the population for 1910.

3. The standings of the champion batters from 1900–1907, inclusive, are here given in percents, for

The National League

38.4 38.2 36.7 35.5 34.9 37.7 33.9 35.

The American League

38.7 42.2 37.6 35.5 38.1 32.9 35.8 35

Graph these percents for each league to a convenient scale, both on the same sheet. Tell what the percent-line shows.

4. The monthly average rainfall or snowfall, in inches, at a certain place for 30 years is as follows:

Jan.	2.8	May	3.59	Sept.	2.91
Feb.	2.30	June	3.79	Oct.	2.63
March	2.56	July	3.61	Nov.	2.66
April	2.70	Aug.	2.83	Dec.	2.71.

Graph these data and tell what the connecting line shows.

5. Graph these average lengths of day from sunrise to sunset in latitude 42°.

		Hr.			Hr.			Hr.
Jan.	16	9.5	May	16	14.5	Sept.	15	12.5
Feb.	15	10.5	June	15	15.0	Oct.	16	11.2
March	16	11.9	July	16	14.9	Nov.	15	9.6
April	15	13.3	Aug.	16	13.9	Dec.	16	9.1.

6. Using the same sheet, scale, and dates as in problem 5, graph the average day's lengths

in latitude 38°: 9.7 10.8 12.0 13.3 14.4 14.9 14.6 13.7 12.5 11.2 10.5 9.5

in latitude 45°: 9.1 10.4 11.9 13.5 14.9 15.6 15.3 14.1 12.6 11.1 9.6 8.8.

What differences in the change of the day's length in different latitudes do the three graphs show?

7. A ship's latitude from week to week was $+42°$, $+38°$, $+30°$, $+20°$, $+12°$, $+2°$, $-1°$, $-6°$, $-3°$, $+12°$. Graph these latitudes and tell when the ship crossed the equator.

Graphing Precise Laws

1. Rectangles are 3 inches wide and their lengths are, 4″ 5″, 6″, 7″, 8″, 9″, 10″, 11″, 12″, 13″. Calculate the areas of these rectangles and plot them to a convenient scale. How does the graph show the area to vary with the base?

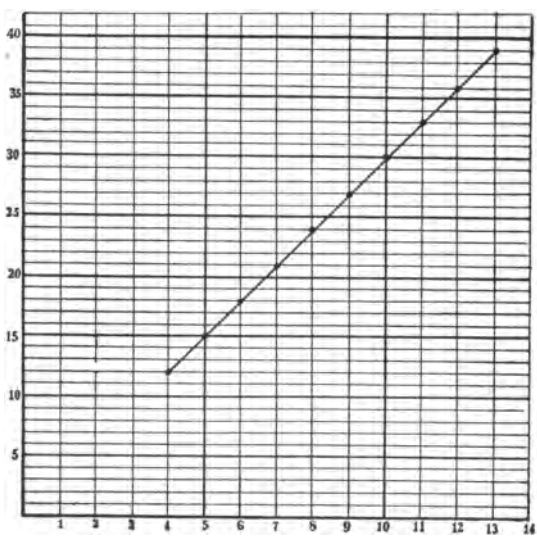


FIG. 69

2. Write the equation for the area, y, of a rectangle having an altitude 3 and a base x.

3. The base of a rectangle is 4 and the altitude is x. Write an equation showing what the area, y, must be.

4. In the equation of problem 3, let x have the values in the first line just below, and calculate the values of y.

$x=1, 2, 3, 4, 5, 6, 7, 8, 9, 10$
$y=4$, etc.

Graph these values of x and y and state how the area, y, varies with the altitude, if the base is 4, a fixed number.

5. The side of a square is x. Write an equation showing how to find its area, y.

6. In the equation of problem 5, suppose x to have the successive values of the first line shown here, and calculate the areas, y.

$x=1, 2, 3, 4, 5, 6, 7$
$y=1, 4$, etc.

7. Graph the values of y (problem 6) on the 1, 2, 3, 4, etc., lines and draw, free-hand, a smooth curve through the points.

8. The altitude of a triangle is 4 and the base is x. Write an equation to show its area, y.

9. Plot the values of y (problem 8) for the values of $x=1$, 2, 3, 4, 5, etc., and tell how the area of a triangle depends on its base if the altitude is a fixed number.

10. Write an equation to show the area, y, of a triangle having a base 6, and an altitude x.

11. Graph the values of y (problem 10) for $x=1$, 2, 3, 4, 5, 6, etc., and show how the area of a triangle depends on its altitude, if its base is a fixed number.

49. We have graphed the laws $y=3x$, $y=4x$, $y=x^2$, and $y=2x$. Other equations may be graphed just as these laws were.

Graph the following equations:

1. $y=x$	4. $y=2x-1$	7. $y=x^2+2$
2. $y=x+1$	5. $y=2x+1$	8. $y=5x$
3. $y=x-1$	6. $y=2x+3$	9. $y=5x-6$.

50. It is not necessary to have the whole equation to make the graph. Only the second member is necessary.

1. Graph $x+3$.

This means take for x the successive values, 1, 2, 3, 4, 5, etc., calculate the values of $x+3$; viz., 4, 5, 6, 7, 8, etc., and plot as before.

2. Graph the following expressions:

(1) $2x$	(4) $2x+1$	(7) $2x-1$
(2) $x+2$	(5) $x+1$	(8) x^2
(3) $x-2$	(6) $x-1$	(9) $4x+2$.

Adding Positive and Negative Numbers

1. Denoting distances traveled northward by positive numbers, and distances traveled southward by negative numbers, find for each of the following cases the distances and the direction of the stopping-point from the starting-point. When the stopping-point is north of the starting-point mark the result +; when south, mark the result −.

An automobile goes:

(1) +15 mi., then −10 mi.	(6) −12 mi., then −10 mi.
(2) +15 mi., " −14 mi.	(7) −20 mi., " +15 mi.
(3) +15 mi., " −20 mi.	(8) −15 mi., " +22 mi.
(4) +25 mi., " −35 mi.	(9) −20 mi., " +21 mi.
(5) −18 mi., " +24 mi.	(10) −22 mi., " +22 mi.

2. In the following problems the numbers indicate distances traveled northward, if negative; and southward, if positive. The *sum in all cases must denote the distance and direction* of the stopping-point from the starting-point. Write the sums with their proper signs:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
+15	−15	+15	−15	+38	−38	+38	−38
+ 8	− 8	− 8	+ 8	+19	−19	−19	+19

(9)	(10)	(11)	(12)	(13)
+ 4	− 4	+11	− 4	+12
+26	−26	−26	+26	−12.

3. Examine (1), (2), (5), (6), (10) of problem 2 and make a rule for adding two numbers having like signs.

4. From (3), (4), (8), (11), (12), and (13) of problem 2 make a rule for adding two numbers having *unlike* signs.

51. Sums, with their proper signs, of positive and negative numbers, are called **algebraic sums.** The sums and differences of numbers regardless of sign, are called **arithmetical sums and differences.**

52. *The algebraic sum of two numbers with like signs is their arithmetical sum, with the common sign prefixed.*

53. *The algebraic sum of two numbers with unlike signs is their arithmetical difference, with the sign of the larger number prefixed.*

1. In the following problems the positive numbers indicate gains and the negative numbers indicate losses. The sums indicate the net change in the man's capital, and whether the net change is an increase or a decrease. Find the sums and tell their meaning:

(1)	(2)	(3)	(4)	(5)	(6)	(7)
$+50$	$+35$	-45	$+75$	-236	$+8x$	$-14a$
$+25$	-38	-20	$+13$	$+780$	$-6x$	$-46a$
-18	$+24$	$+60$	-86	-95	$-4x$	$+77a$
-6	-15	$+55$	$+8$	$+45$	$+7x$	$-5a.$

2. State a way of adding any number of positive and negative numbers.

3. A force of 12 lb. pulling toward the right (+12 lb.) together with a force of 9 lb. pulling toward the left give a combined pull equal to what force?

4. What single force has the same effect in pulling the ring R as the following pairs of forces acting together?

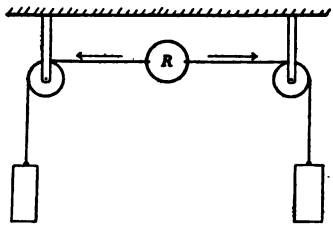


FIG. 70

(1) $+12$ lb. and -8 lb.	(9) -16 lb. and -8 lb.
(2) -12 lb. " $+8$ lb.	(10) $+3$ lb. " -12 lb.
(3) -10 lb. " $+10$ lb.	(11) $+x$ lb. " $+y$ lb.
(4) -16 lb. " $+13$ lb.	(12) $+x$ lb. " $-y$ lb.
(5) $+14$ lb. " -17 lb.	(13) $-x$ lb. " $+y$ lb.
(6) $+9$ lb. " -20 lb.	(14) $-x$ lb. " $-y$ lb.
(7) $+11$ lb. " $+15$ lb.	(15) $+x$ lb. " $-x$ lb.
(8) -16 lb. " $+12$ lb.	(16) $-2x$ lb. " $+x$ lb.

5. A man draws a pail of brick, weighing 60 lb., to a house-top by pulling on a rope which runs over a pulley, with a force of 65 lb. What single force equals the sum of the two forces acting on the handle of the pail?

6. A balloon pulls upward on a stone, weighing 6 oz., with a force of 8 oz. What is the sum of the forces?

7. A piece of iron weighing 18 lb., when placed under water, is pushed (buoyed) upward with a force of $2\frac{4}{7}$ pounds. What is the sum (combined effect) of the two forces together?

8. An elevator starts at a certain floor, goes up 65 ft., down 91 ft., up 52 ft., down 13 ft., and up 65 ft., and stops. How far and in what direction is the stopping-point from the starting-point? Give your answer in the form of an algebraic sum.

9. A vessel starting in latitude $+20°$ sails $+13°$ in latitude, then $-60°$, then $+40°$, then $-10°$. What is its latitude after the sailings? What is the latitude of a ship starting in latitude $-50°$ after these changes of latitude: $+10°$, $-5°$, $+18°$, $-7°$, $+38°$, $-12°$, $+60°$?

10. A boatman rows, at a rate that would carry him 3 miles an hour through still water, down a river whose current is 2 mi. an hour. What is his rate per hour? What is his rate per hour, if he rows up the river?

Exercise IX

Add the following, doing all you can orally:

1. $+\frac{3}{4}$	7. $-6\frac{2}{3}$	13. $-13\frac{1}{2}$	19. $+32b^2c$
$+\frac{5}{8}$	$+8\frac{5}{6}$	$+23\frac{3}{8}$	$-28b^2c$
2. $-\frac{7}{8}$	8. $+7\frac{3}{5}$	14. -6.69	20. $+18v^2y^3$
$+\frac{3}{4}$	$-7\frac{4}{5}$	$+8.04$	$-24v^2y^3$
3. $+\frac{11}{16}$	9. $-12\frac{1}{2}$	15. $+8.95$	21. $+8(x-a)$
$-\frac{5}{6}$	$-2\frac{1}{2}$	-11.25	$-6(x-a)$
4. $+3\frac{1}{2}$	10. $-18\frac{1}{3}$	16. $-16r$	22. $-12(x+y)$
$-2\frac{1}{3}$	$+26\frac{1}{4}$	$+18r$	$-7(x+y)$
5. $-5\frac{1}{4}$	11. -2.12	17. $-3.2s$	23. $-68(m-r)$
$-2\frac{3}{8}$	-1.88	$-6.8s$	$-75(m-r)$
6. $+4\frac{3}{4}$	12. $+3.16$	18. $+7\frac{2}{3}x$	24. $-6\frac{1}{2}(c-d)$
$-6\frac{1}{8}$	-4.08	$-6\frac{5}{6}x$	$+3\frac{2}{3}(c-d)$

Subtracting Positive and Negative Numbers

1. In this problem positive numbers indicate the readings above zero and negative numbers, readings below zero. The *difference means* the number of degrees the top of the mercury column must rise, or fall, to change from the second reading

to the first. If the change is a *rise*, mark it positive (+), if a *fall*, mark it negative (−).

	(1)	(2)	(3)	(4)	(5)
First reading:	$+68^\circ$	-98°	-30°	-7°	$+1^\circ$
Second reading:	$+42^\circ$	-18°	$+65^\circ$	$+32^\circ$	-28°

	(6)	(7)	(8)	(9)	(10)
First reading:	0°	-8°	$+6x^\circ$	$-4a^\circ$	$-2y^\circ$
Second reading:	$+78^\circ$	$+8^\circ$	$-3x^\circ$	$-9a^\circ$	$+60y^\circ$

Define *minuend.* Define *subtrahend.*

2. By sections **52** and **53** find the sums in the following problems and compare the exercises and your results with those of the like numbered exercises of problem 1:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
$+68$	-98	-30	-7	$+1$	0	-8	$+6x$	$-4a$	$-2y$
-42	$+18$	-65	-32	$+28$	-78	-8	$+3x$	$+9a$	$-60y$

3. Show by comparing the problems of 1 and 2 that the difference of any two numbers can be found by changing the sign of the subtrahend and then adding.

4. Find the differences of the following:

(1)	(2)	(3)	(4)	(5)	(6)	(7)
$+19$	-60	-75	$+8$	$-3a$	$+18x$	$-12y$
-10	-25	$+25$	-16	$-2a$	$-6x$	$-7y$

(8)	(9)	(10)	(11)
$+a$	$+3x-18$	$+a+2b-c$	$-12a+3x$
$-b$	$+2x+6$	$+3a-5b+3c$	$+7a-2x$

54. *To find the difference of two numbers change the sign of the subtrahend mentally, then add.*

Exercise X

Subtract the lower from the upper number of the following, doing all you can mentally:

1. $+\frac{7}{8}$	9. $+12\frac{1}{8}$	17. $-9(c-s)$
$\underline{+\frac{3}{4}}$	$\underline{-16\frac{1}{2}}$	$\underline{+3(c-s)}$
2. $-\frac{5}{8}$	10. $-11\frac{1}{9}$	18. $+ac^4d^3s$
$\underline{-\frac{1}{4}}$	$\underline{+\ 8\frac{2}{3}}$	$\underline{-ac^4d^3s}$
3. $+\frac{7}{9}$	11. $-6av^2$	19. $+6.75a^3$
$\underline{-\frac{2}{3}}$	$\underline{+\ av^2}$	$\underline{-7.25a^3}$
4. $+\frac{6}{7}$	12. $-13bc^3$	20. $-3.16c^2d$
$\underline{-\frac{2}{5}}$	$\underline{+\ 8bc^3}$	$\underline{-0.89c^2d}$
5. $-\frac{9}{10}$	13. $-\ 9cxy$	21. $-4.76(t+s)$
$\underline{+\ \frac{2}{3}}$	$\underline{-15cxy}$	$\underline{+9.67(t+s)}$
6. $+4\frac{1}{3}$	14. $+7ab^2c$	22. $+0.82(a^2-x^2)$
$\underline{-5\frac{1}{4}}$	$\underline{-9ab^2c}$	$\underline{-3.75(a^2-x^2)}$
7. $-8\frac{2}{3}$	15. $-8m^3r^2s$	23. $-0.75(c+d^2)$
$\underline{+2\frac{1}{2}}$	$\underline{+3m^3r^2s}$	$\underline{-0.90(c+d^2)}$
8. $-9\frac{2}{5}$	16. $+6(x+z)$	24. $+0.916ab^2c^3$
$\underline{-3\frac{1}{3}}$	$\underline{-7(x+z)}$	$\underline{-1.803ab^2c^3}$

Multiplying Positive and Negative Numbers

Suppose the short spaces on the line east-west, E W, represent a mile.

1. Show what space starting from 0 in each case represents +2 mi.; +3 mi.; +5 mi.; +8 mi.; +10 mi.

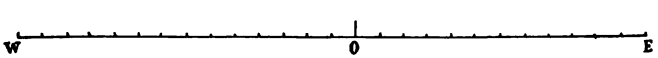


Fig. 71

2. Starting again from 0, show the space that represents −1 mi.; −2 mi.; −5 mi.; −7 mi.; −10 mi.

3. Show the spaces from 0 a man goes if he travels $+2$ mi. a day for 1 day; 2 days; 3 days; 5 days.

4. What is 2 times -2 mi.? 3 times -2 mi.? 4 times -2 mi.? 5 times -2 mi.?

5. What is 5 times -4 mi.? -10 mi.? -20 mi.? -25 mi.? -100 mi.?

6. What is 16 times -2 mi.? -5 mi.? -10 mi.? -20 mi.? -100 mi.?

7. What is $50\times(-2)$? $25\times(-10)$? $40\times(-12)$? $8\times(-120)$? $4\times(-a)$? $10\times(-x)$?

8. The value of a man's property changes by $+\$1000$ a year. How much does it change in 2 yr.? 5 yr.? 8 yr.? 10 yr.? In each case tell whether the change is an increase, or a decrease.

9. If the value of a man's property changes by $-\$500$ a year, how much does it change in 3 yr.? 5 yr.? 7 yr.? 8 yr.? 10 yr.? 12 yr.? In each case tell whether the change is an increase, or a decrease.

10. How much and in what way does a man's property change in 12 yr. at the rate of $+\$50$ a yr.? $-\$100$ a yr.? $+\$400$ a yr.? $+\$1,800$ a yr.? $-\$2,000$ a year?

11. How much and in what direction does the height of a mercury column of a thermometer change in 6 hours at the rate of $+10°$ an hr.? $-8°$ an hr.? $-7°$ an hr.? $+5\frac{1}{2}°$ an hr.? $-3\frac{1}{2}°$ an hr.? $+2\frac{1}{2}°$ an hr.? $-a°$ an hour?

12. State a way of multiplying a negative number by an arithmetical number.

55. A light bar (Fig. 72) supplied with equally spaced pegs is balanced about its middle point, M. With a number of equal weights, w, the following experiments are performed.

1. Hang a weight of $2w$ on the peg l_1. This weight tends to turn the left end of the bar downward. How much weight must be attached to the hook, H, to balance this turning-tendency? Now hang the same weight, $2w$, on peg l_2, and measure its downward turning-tendency by attaching to the hook, H, a weight sufficient to balance the bar.

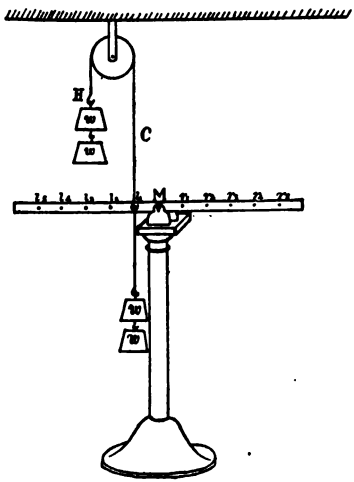


FIG. 72

2. In a similar manner find the turning-tendency caused by the weight $2w$ on the peg l_3; on l_4; on l_5.

3. Using a weight of $3w$, find the turning-tendency when placed on peg l_1; on l_2; on l_3; on l_4; on l_5.

4. Perform experiment 3, using a weight $4w$.

56. Experiments 1, 2, 3, and 4 show that when a weight (as $2w$) is hung on peg l_5, the turning-tendency caused by this weight ($2w$) is five times as great as when the same weight is hung on peg l_1; on l_4 its turning-tendency is four times as great as on l_1; on l_3 it is three times, and on l_2 it is two times as great as on l_1.

57. The same facts hold when $3w$, $4w$, or any other weight is used. In other words, with an apparatus like that in Fig. 72, *the turning-tendency, or leverage, caused by a weight is measured by the product of the weight by the distance from the turning-point, M, to the peg where the weight hangs.*

1. What is the turning-tendency caused by a weight of $7w$ hung on peg l_5? on l_3? a weight of $9w$ on l_4? on l_1?

2. If the bar had 12 pegs on each side of M, what would be the turning-tendency caused by a weight of $8w$ on peg l_{12}? $7w$ on l_9? w on l_{11}?

3. Attach the cord, C (Fig. 72), to peg r_1 and perform the experiments of §55 on the right side of the bar.

58. Experiment 3, §57, shows that the same facts are true on the right side as on the left, but the bar turns, or tends to turn, in the *opposite direction.* To avoid confusion, some simple method of distinguishing between these two directions of turning is desired.

Suppose a watch laid upon the page of the book with its face up. When the bar turns, or tends to turn *with* the hands of the watch, the turning-tendency will be called *negative* and designated $-$; if it turns, or tends to turn, *against* (opposite to) the watch-hands, the turning-tendency will be called *positive* and designated $+$.

59. In all the experiments thus far performed the weights which caused the bar to turn were downward-pulling weights, or forces. By arranging an apparatus as in Fig. 73, p. 82, forces can also be made to pull upward. Downward-pulling weights or forces will be designated by $-$, and upward-pulling forces by $+$.

60. The distance from the turning-point, M, to the peg where the weight, or force, acts will be called the *lever-arm,* or *arm,* of the force. Lever-arms measured from the turning-point toward the right will be marked $+$; those toward the left, $-$.

For example, if the distance from M to peg r_1 be represented by $+1$, then the distance from M to the peg r_4 will be represented by $+4$; from M to l_3, by -3, and so on.

61. If a force of -2 acts on peg l_3, its turning-tendency, or leverage, is the product of -2 and -3. Since the bar tends to turn *against* the watch-hands, the turning-tendency is written $+6$. If T represents the turning-tendency, then

$$T=(-2)\cdot(-3)=+6.$$

If a force of -2 acts on r_3, its turning-tendency is the product of -2 and $+3$, or -6, since the bar tends to turn *with* the watch-hands. In this case

$$T=(-2)\cdot(+3)=-6.$$

1. In the following experiments write (as in §**61**) the leverages, or turning-tendencies, for the forces and arms indicated:

	Forces	Arms	Leverages		Forces	Arms	Leverages
I	-3	-2		VII	-6	$+1$	
II	-4	$+2$		VIII	-8	-1	
III	-7	-3		IX	$-a$	$+b$	
IV	-6	-4		X	$-m$	$-z$	
V	-1	$+5$		XI	$-3w$	$-2x$	
VI	-8	$+9$		XII	$-6c$	$-12y$	

2. If loadings I and VII in the foregoing tables were on the apparatus at the same time, would the bar balance or turn? If it turns, in what direction would it turn? Answer similar questions for II and VIII; V and VII; VI and VIII.

62. By the apparatus in Fig. 73 (p. 82) forces can be made to pull upward on either side of the bar.

If a force of 2 pulls upward on peg r_4, its turning-tendency is the product of $+2$ and $+4$. Since the bar tends to turn *against* the watch-hands, the turning-tendency is written $+8$. Using T for the turning-tendency,

$$T=(+2)(+4)=+8.$$

On peg l_4 the turning-tendency is the product of $+2$ and -4, or -8, since the bar tends to turn *with* the watch hands. In this case

$$T=(+2)(-4)=-8.$$

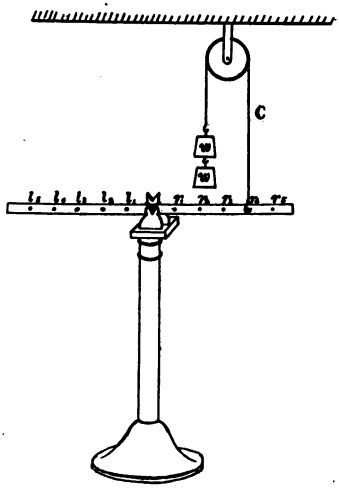


FIG. 73

1. In the following experiments write the leverages for the indicated arms and forces:

	Forces	Arms	Leverages		Forces	Arms	Leverages
I	$+3$	-2		V	$+6$	$+1$	
II	$+7$	-3		VI	$+m$	$+7a$	
III	$+1$	$+3$		VII	$+3$	-2	
IV	$+8$	$+9$		VIII	$+6w$	$-12z$	

2. If III and VII in the foregoing tables are on the apparatus at the same time, will the bar balance or turn? If it turns, in which direction will it turn? Answer similar questions for I and V; II and VII; IV and VII.

3. Write the leverages for the following loadings:

	Forces	Arms		Forces	Arms
I	$+8$	-9	VII	$+7$	$+8$
II	$+8$	$+9$	VIII	-7	$+8$
III	-8	$+9$	IX	$+7$	-8
IV	-8	-9	X	-7	-8
V	$+x$	$-7y$	XI	$-a$	$+6b$
VI	$+4x$	$+3y$	XII	$-5a$	$-2b$

4. What sign (direction) has the turning-tendency of a plus force with a plus arm? A plus force with a minus arm? A minus force with a plus arm? A minus force with a minus arm?

63. If $3(-4)$ means that four is to be measured, or laid

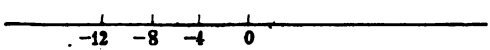


FIG. 74

off, 3 times from 0 in the *negative direction* (the direction of -4), what is the value of the product, $3(-4)$, or, what is the same thing, $(+3)(-4)$?

The product means the distance and direction from 0, evidently -12.

1. Show on a figure the meaning and value of the following products:

(1) $2(+3)$ (3) $3(-2)$ (5) $3(-5)$ (7) $2(+6)$
(2) $2(-3)$ (4) $3(+2)$ (6) $1(+5)$ (8) $2(-6)$.

2. Show on a figure the products: $(-3)(+4)$; $(-3)(-4)$.

$(-3)(+4)$ means that four is to be laid off 3 times in the direction *opposite* to the direction of $+4$, i. e., *in the negative direction.*

$(-3)(-4)$ means that four is to be laid off 3 times in the direction *opposite* to the direction of -4, i. e., *in the positive direction.*

3. Interpret these products on the same principle:

(1) $(-2)(-3)$	(4) $(+2)(+8)$	(7) $(-2)(+5)$
(2) $(+3)(-2)$	(5) $(-3)(-5)$	(8) $(-2)(-5)$
(3) $(-2)(+4)$	(6) $(+3)(-5)$	(9) $(-3)(+6)$.

4. Show on Fig. 75 the value of 2 times $+a$; $2(-a)$; $(-2)(+a)$; $(-2)(-a)$; $(-3)(-a)$; $(-3)(+a)$; $(-4)(-a)$.

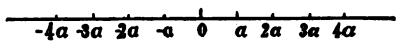


FIG. 75

5. Interpret the meanings of:

(1) $(+3)(+a)$	(5) $(+a)(+b)$	(9) $(+c)(-d)$
(2) $(+3)(-a)$	(6) $(+a)(-b)$	(10) $(+c)(+d)$
(3) $(-3)(+x)$	(7) $(-a)(+b)$	(11) $(-c)(+d)$
(4) $(-3)(-x)$	(8) $(-a)(-b)$	(12) $(-c)(-d)$.

SUMMARY

In digits:	*In letters:*
$(+3)(+4)=+12$	$(+a)(+b)=+ab$
$(+3)(-4)=-12$	$(+a)(-b)=-ab$
$(-3)(+4)=-12$	$(-a)(+b)=-ab$
$(-3)(-4)=+12$	$(-a)(-b)=+ab$.

6. Examine the eight products of the summary and make a rule for obtaining the algebraic sign of a product of two numbers from the signs of the factors.

Compare your rule with

64. Law of Signs for Multiplication:

If two factors have **like** *signs, the product is positive; if two factors have* **unlike** *signs, the product is negative.*

EXERCISE XI

Find the products of the following, doing all you can mentally:

1. $(+\frac{2}{3})(+\frac{1}{4})$
2. $(-\frac{2}{3})(+\frac{5}{6})$
3. $(+\frac{3}{4})(-\frac{3}{8})$
4. $(-\frac{3}{4})(+\frac{5}{9})$
5. $(-\frac{7}{8})(-\frac{3}{14})$
6. $(+\frac{7}{11})(-\frac{5}{6})$
7. $(-6\frac{3}{4})(-6\frac{3}{4})$
8. $(+6\frac{3}{4})(-6\frac{3}{4})$
9. $(-6\frac{3}{4})(+6\frac{3}{4})$
10. $(+6\frac{3}{4})(+6\frac{3}{4})$
11. $(+3)(-ad)$
12. $(-3)(-ax)$
13. $(-7)(+rs)$
14. $(-a)(+pq)$
15. $(-c)(-pr)$
16. $(-3\frac{4}{5})(+xy)$
17. $(-5\frac{1}{6})(-xz)$
18. $(-5\frac{1}{2}a)(+b)$
19. $(-2\frac{1}{2}c)(-2\frac{1}{2}d)$
20. $(+7\frac{1}{3}x)(-7\frac{1}{3}y)$
21. $(+5)(-gl^2)$
22. $(-7)(-g^2l)$
23. $(-10)(+v^2x)$
24. $(+12)(-ay^3)$
25. $(-16)(+abc^2)$
26. $(+bc)(-a)$
27. $(+v^2x)(-a^2b)$
28. $(+x^2)(-x^3)$
29. $(+y^3)(-x^2)$
30. $(+6r^2)(-\frac{1}{2}r^2)$.

Dividing Positive and Negative Numbers

65. The product of two factors and either factor being known, the other factor is the quotient arising, as in arithmetic, from dividing the known product by the known factor.

66. The quotient of ab divided by b is indicated thus, $ab \div b$, *or* $\frac{ab}{b}$; of ab divided by $-a$, thus, $ab \div (-a)$, or $\frac{ab}{-a}$.

1. Since $(+2)(-5) = -10$, what must $-10 \div (+2)$, or $\frac{-10}{+2}$ be? $\frac{-10}{-5}$?

2. Looking at the first four equations in the foregoing summary, §§**63** and **64**, answer the following questions, giving reasons for answers:

(1) $(+12) \div (+3) = ?$
(2) $(+12) \div (+4) = ?$
(3) $(-12) \div (+3) = ?$
(4) $(-12) \div (-4) = ?$
(5) $(-12) \div (-3) = ?$
(6) $(-12) \div (+4) = ?$
(7) $(+12) \div (-3) = ?$
(8) $(+12) \div (-4) = ?$

3. From the second list of four equations in the foregoing summary answer, with reasons, the following:

(1) $(+ab)\div(+a)=?$ (5) $(-ab)\div(-a)=?$
(2) $(+ab)\div(+b)=?$ (6) $(-ab)\div(+b)=?$
(3) $(-ab)\div(+a)=?$ (7) $(+ab)\div(-a)=?$
(4) $(-ab)\div(-b)=?$ (8) $(+ab)\div(-b)=?$

4. Answer the following, with reasons:

(1) $\frac{+18}{+2}=?$ (5) $\frac{+18}{+9}=?$ (9) $\frac{xy}{y}=?$

(2) $\frac{+18}{-2}=?$ (6) $\frac{+18}{-9}=?$ (10) $\frac{-xy}{y}=?$

(3) $\frac{-18}{+2}=?$ (7) $\frac{-18}{+9}=?$ (11) $\frac{-xy}{-x}=?$

(4) $\frac{-18}{-2}=?$ (8) $\frac{-18}{-9}=?$ (12) $\frac{-xy}{-y}=?$

5. Examine the answers to problems 2 and 3 and state what the sign of the quotient is

(1) if the sign of **both** dividend and divisor is **plus,** (+);

(2) if the sign of **both** dividend and divisor is **minus,** (−);

(3) if the sign of the dividend is **plus,** (+), and of the divisor, **minus,** (−);

(4) if the sign of the dividend is **minus,** (−), and of the divisor, **plus,** (+);

(5) if the signs of dividend and divisor are **alike** (i. e., both +, or both −);

(6) if the signs of dividend and divisor are **unlike** (i. e., one − and the other +).

6. State the law of signs for division and compare the statement with

67. Law of Signs for Division:

If dividend and divisor have **like** *signs the quotient is positive; if dividend and divisor have* **unlike** *signs the quotient is negative.*

Answer the following:

1. $\frac{+18}{-6} = ?$
2. $\frac{-25}{-5} = ?$
3. $\frac{-30}{+10} = ?$
4. $\frac{a}{a} = ?$
5. $\frac{a^2}{-a} = ?$
6. $\frac{-mp}{m} = ?$
7. $\frac{6a}{-6} = ?$
8. $\frac{-6a}{a} = ?$
9. $\frac{-12ax}{12a} = ?$

Exercise XII

Find the quotients of the following, doing all you can mentally:

1. $(+\frac{3}{4}) \div (+\frac{1}{3})$
2. $(-\frac{7}{8}) \div (+\frac{3}{4})$
3. $(+\frac{7}{8}) \div (-\frac{3}{4})$
4. $(+\frac{5}{7}) \div (+\frac{2}{7})$
5. $(+\frac{5}{7}) \div (-\frac{2}{7})$
6. $(+\frac{5}{6}) \div (-\frac{2}{3})$
7. $(-\frac{5}{6}) \div (+\frac{3}{5})$
8. $(+\frac{5}{7}) \div (-\frac{3}{4})$
9. $(-\frac{5}{7}) \div (-\frac{3}{4})$
10. $(-\frac{7}{8}) \div (+\frac{5}{6})$
11. $(-\frac{5}{6}) \div (-\frac{3}{7})$
12. $(+\frac{3}{7}) \div (+\frac{5}{6})$
13. $(-\frac{3}{7}) \div (-\frac{5}{6})$
14. $(-\frac{5}{6}) \div (+\frac{3}{7})$
15. $(+\frac{11}{12}) \div (-\frac{3}{5})$
16. $(-2a) \div (+2)$
17. $(-2a) \div (-a)$
18. $(-5b) \div (+5)$
19. $(+12x) \div (-4x)$
20. $(-a) \div (-\frac{1}{2}a)$
21. $(+a) \div (-\frac{1}{4}a)$
22. $(-a^3) \div (+a)$
23. $(+a^2) \div (-a^2)$
24. $(-a^4) \div (+a^3)$
25. $(-6a^3) \div (+3a)$
26. $(+4ab) \div (-b)$
27. $(+4ab) \div (-2a)$
28. $(-3ax) \div (+3x)$
29. $(-2xy) \div (+xy)$
30. $(+6\frac{2}{3}r^2) \div (-3\frac{1}{4}r)$
31. $(+av^3) \div (-av)$
32. $(-av^3) \div (+v^2)$
33. $(-abc) \div (-a)$
34. $(+abc) \div (+ac)$
35. $(-abc^2) \div (-ac)$
36. $(-avr) \div (+ar)$
37. $(+7\frac{2}{3}ax) \div (+23a)$
38. $(-7\frac{2}{3}yz) \div (-11\frac{1}{2}y)$
39. $(-16\frac{1}{2}yz^3) \div (+4\frac{1}{8}yz)$
40. $(+6.82az) \div (-31a)$.

Summary

Positive and **negative numbers** are needed to record temperature, directed distances, opposing forces, opposite latitudes, debts and assets, dates before and after the beginning of the Christian era, motions in opposite directions, and directed magnitudes.

Graphs may be made to represent thermometer readings, average temperatures, ages and heights of persons, batting averages, populations, rain and snowfall, lengths of day, areas of figures, equations, and general expressions of number.

The **algebraic sum** of numbers with *like* signs is the arithmetical sum, with the common sign prefixed.

The **algebraic sum** of two numbers with *unlike* signs is the *arithmetical* difference, with the sign of the larger number prefixed.

The **algebraic sum** of any number of numbers may be found (1) by adding all the numbers in order, or (2) by adding all the positive numbers, then all the negative numbers, then adding the sums.

The **algebraic difference** of two numbers is found by adding to the minuend the subtrahend *with sign changed.*

The turning-tendency, or leverage, of a force on a bar or lever, is the product of the force by the lever-arm.

If two factors have *like* signs the **product** is *positive.* If two factors have *unlike* signs the **product** is *negative.*

If dividend and divisor have *like* signs the **quotient** is *positive.*

If dividend and divisor have *unlike* signs the **quotient** is *negative.*

CHAPTER V

BEAM PROBLEMS IN ONE AND TWO UNKNOWNS

Problems in One Unknown Number

68. In this chapter some practical problems arising out of the common uses of forces will be solved by means of the equation. It is necessary first to discover a law of these forces.*

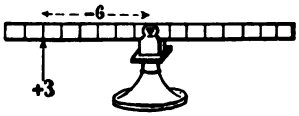


FIG. 76

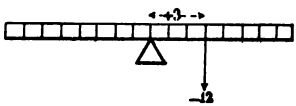


FIG. 77

1. A bar or lever (Fig. 76) has loadings as in Fig. 78. Draw a diagram (Fig. 77) and find the turning-tendency, or leverage, for each loading:

No.	Force	Arm	No.	Force	Arm
I	+3	−6	VII	− 4	−9
II	+2	−3	VIII	− 3	−2
III	+3	+2	IX	−12	+3
IV	x	+3	X	x	−3
V	x	−2	XI	x	+2
VI	+3	x	XII	− 2	x

FIG. 78

2. Draw a diagram showing I and VIII (Fig. 78) on the bar at the same time; I and VII; II and VIII; II and IX;

* See pages 78–83, on turning-tendencies.

I, III, and VIII. In each case state whether the bar balances or turns, and if it turns, in which direction it turns.

69. In problem 2, if the turning-tendencies for I and VIII are added, the *total* turning-tendency is

$$(+3)\cdot(-6)+(-3)\cdot(-2)=(-18)+(+6)=-12,$$

which says in mathematical language that the bar does not balance, but turns in the negative direction.

If the turning-tendencies for II and VIII are added, the *total* turning-tendency is

$$(+2)\cdot(-3)+(-3)\cdot(-2)=(-6)+(+6)=0,$$

which shows that the bar does not turn.

If the turning-tendencies for III, VII, and IX are added, the *total* turning-tendency is

$$(+3)(+2)(-4)(-9)+(-12)(+3)=+6+36-36=+6,$$

which shows that the bar does not balance, but turns in the positive direction.

70. If two or more forces are acting on the bar at the same time, the *total turning-tendency* is found by *adding* algebraically the separate turning-tendencies. If the *algebraic sum* is zero, the bar balances. If the sum is not zero, the bar turns in the direction indicated by the sign of the sum.

1. Find the total turning-tendency, and interpret it, as in §69, when the following loadings of Fig. 78 are on the bar at the same time. Draw a diagram for each case:

(1) I and IX
(2) II and VII
(3) I, III, and IX
(4) I, III, and VII.

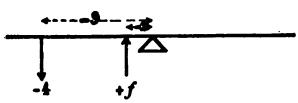


FIG. 79

2. If a force of -4 on an arm of -9 (Fig. 79), and a force of $+f$ on an arm of -2, are on the bar, what must f be, for balance?

If the bar is balanced, the sum of the turning-tendencies must be zero. We may then write

$$(f)(-2)+(-4)(-9)=0. \qquad (1)$$

Multiplying $\quad -2f+36=0. \qquad (2)$

Subtracting 36 $\quad -2f=-36.$ What axiom? $\qquad (3)$

Dividing by -2 $\quad f=\ \ 18.$ What axiom? $\qquad (4)$

CHECK: From equation (1), $(+18)(-2)+(-4)(-9)=(-36)+(+36)=0.$

3. Write the equation of total turning-tendency, and find what the unknown force or arm must be, for balance, for each of the five loadings of Fig. 80.

No.	Force	Arm	Force	Arm	Force	Arm
I	f	$+3$	$+3$	-6		
II	$+3$	-6	w	-2		
III	r	-2	r	-3	-4	-9
IV	-2	d	d	$+3$	$+2$	-3
V	-2	l	$+3$	l	$+3$	$+2$

FIG. 80

71. Law of Turning-Tendencies or Leverages. *For balance, the algebraic sum of all the turning-tendencies must equal zero.*

1. A bar (Fig. 81) is balanced by a force of $+10$ on an arm of -6, and a force of $s+3$ on an arm of $+5$. Find the values of s and $s+3$.

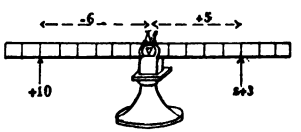


FIG. 81

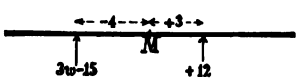


FIG. 82

2. A force of $3w-15$ (Fig. 82) on an arm of -4 is balanced by a force of $+12$ on an arm of $+3$. Find the values of w and $3w-15$.

3. A bar is balanced by each of the five loadings of Fig. 83. Draw a sketch and find the value of the unknown forces and arms for each loading.

No.	Force	Arm	Force	Arm	Force	Arm
I	$w+5$	-3	$+15$	$+4$		
II	$t-3$	$+7$	$-t$	-8	$+13$	-3
III	$+3$	$m-5$	$+15$	-4		
IV	$+3$	-8	$+3k$	$+4$	-2	$+4k$
V	$+39\frac{3}{8}$	-4	$5s$	$+2\frac{1}{2}$	$-13\frac{1}{8}$	-4

FIG. 83

Practical Applications

1. A B (Fig. 84) is a crowbar, $6\frac{1}{2}$ ft. long, supported at F, $\frac{1}{2}$ ft. from A. A stone presses down at A with a force of 1,800 pounds. How many pounds of force must be exerted by a man pressing down at B to raise the stone?

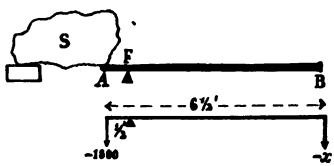


FIG. 84

2. With other conditions as in problem 1, what would be the pressure at B if the fulcrum F (point of support) were 3 in. from A?

3. With the fulcrum $\frac{1}{2}$ ft. from A (Fig. 84) what weight would be held in balance by a pressure of 200 lb. at B?

4. A suction pump (Fig. 85) is a device for raising water from wells. The handle, O B, works against a pin at A, so that when the hand pushes downward at B, the point O rises, and by the aid of a piston on the lower end, C, raises a mass of water. If O A = 2 in. and O B = 3 ft., what load at O will be raised by a force of 20 lb. pushing downward at B?

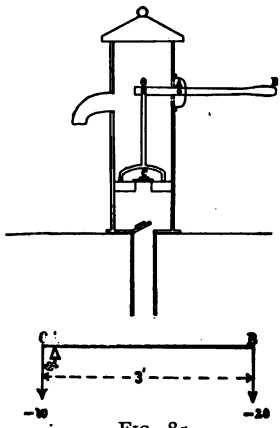


FIG. 85

5. With other conditions as in 4, what force will be exerted at O by a downward force of 68 lb. at B?

6. A stone slab S (Fig. 86), weighing 2,400 lb., rests with its edge on a point B, 6 in. from the fulcrum F of a crowbar F A, 6 ft. long. How many pounds of force must be exerted at A to raise the slab?

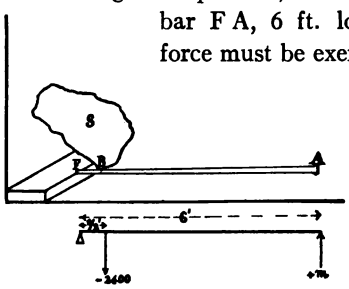


FIG. 86

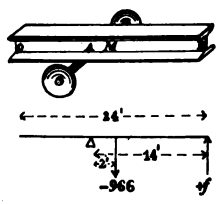


FIG. 87

7. A steel beam 24 ft. long and weighing 966 lb. (Fig. 87) is being moved by placing under it an axle borne by a pair of wheels, as shown at A, the end B being carried. If the axle is 2 ft. from the middle of the beam, what is the weight at B?

The weight of the beam itself may be treated as a load of 966 lb. hanging to the bar at the middle point.

8. If the supporting point in problem 7 had been 11 ft. from O, what force at B would have balanced the rail?

9. How far must the axle in problem 7 be placed from the middle M, that the weight at B may be 241½ pounds (Fig. 88)?

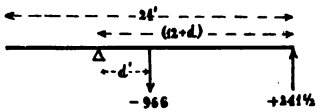


FIG. 88

10. With the fulcrum 2 ft. to the left of M, what would be the weight of the rail if it is balanced by an upward force of 140 lb. at B?

11. How may a steel rail weighing more than a ton be weighed with a pair of balances reading only to 60 pounds?

12. A dry goods box (Fig. 89) weighing 360 lb. is being moved along the floor by the aid of a roller. If the box is 6 ft. long, what force at C is needed to hold the box horizontal when the roller is 1 ft. from B? 2 ft. from B?

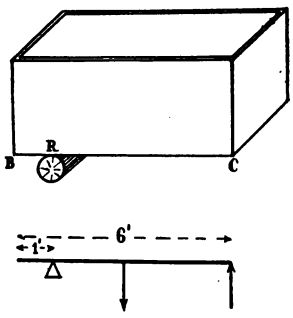


FIG. 89

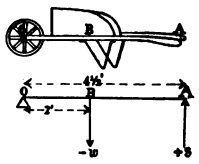


FIG. 90

13. A wheelbarrow (Fig. 90) is loaded with 45 bricks, averaging 6 lb. a piece. What lifting force will be needed at A to raise the load if the bar O A is 4½ ft. and the distance

from the center of the wheel, O, to the point, B, where the vertical line through the center of the load crosses A O, is 2 feet?

14. With the same load and length of bar, O A, as in problem 13, how far is it from O to the crossing-point B of the vertical center line of the load, if 90 lb. at A just raises it?

Exercise XIII

Simplify the following products and quotients:

1. $(-15-l)(+3)$
2. $(10-t)(-85)$
3. $(3l)(7-t)$
4. $(w-4)(+8\frac{1}{2})$
5. $(-15-7t)(-6)$
6. $(-1)(-5x)$
7. $(-1)(-5x+7)$
8. $(+1)(-5x+7)$
9. $(+5)(0)$
10. $(-5)(0)$
11. $(0)(+342)$
12. $(0)(-763)$
13. $(-2k)(l+x)$
14. $\dfrac{-36xy}{+4y}$
15. $\dfrac{-15x}{-1}$
16. $\dfrac{-9y}{+1}$
17. $\dfrac{-15x+7}{-1}$
18. $\dfrac{-15x+7}{+1}$
19. $\dfrac{0}{+7}$
20. $\dfrac{0}{-7}$.

72. It is often convenient to measure lever-arms from a point at which the bar is not actually supported.

1. A basket weighing 56 lb. hangs on a stick 8 ft. long (Fig. 91) at a point 1 ft. from the middle, while it is being

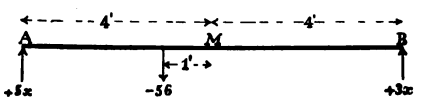


Fig. 91

carried by two boys, one at each end. The boys lift $5x$ and $3x$ lb., respectively. Find the values of x, $5x$, and $3x$.

The bar or stick is balanced, no matter what point on the bar is regarded as the turning-point, or fulcrum. The lever-arms may, therefore, be measured from any point on the bar, whether the bar is actually supported at that point, or not.

Taking M as the turning-point, the lever-arms are

$$-4, \quad -1, \quad +4,$$

and the turning-tendencies are

$$(+5x)(-4), \quad (-56)(-1), \quad (+3x)(+4).$$

Since the bar does not turn, the sum of these turning-tendencies must equal zero.

$$(+5x)(-4)+(-56)(-1)+(+3x)(+4)=0.$$

Solve the equation.

It should be kept in mind that once a certain point is selected from which to measure lever-arms, all lever-arms must be measured from this point throughout the solution.

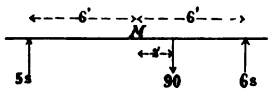


FIG. 92

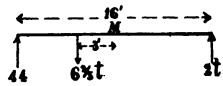


FIG. 93

2. A bar is balanced by the forces shown in Figs. 92 and 93. Find the values of the unknown forces, measuring all lever-arms from M.

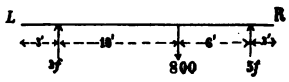


FIG. 94

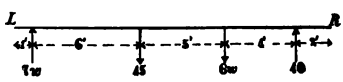


FIG. 95

3. Find the values of the unknown forces on a bar balanced as shown in Figs. 94 and 95. Measure all lever-arms from the end-point L.

4. Solve problem 3, measuring all lever-arms from the end-point R.

Problems in Two Unknowns

73. Beam or lever problems in which *two numbers are unknown* also may be solved by the aid of the law of leverages.

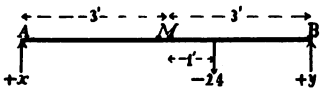


FIG. 96

1. A basket weighing 24 lb. (Fig. 96) hangs on a stick 6 ft. long, at a point 1 ft. from the middle, while it is being carried by two boys, A and B. How much does each boy lift?

Using M as turning-point, $3y-24-3x=0$. Why? (1)

$y=8+x$. Why? (2)

2. Can we find from the foregoing equation (2) the values of x and y that satisfy problem 1? To study this question, copy and fill in the following:

In the equation $y=8+x$ (2)

	x	y
if $x=$ 1, then $y=8+1=9$	1	9
if $x=$ 2, then $y=8+2=10$	2	10
if $x=$ 5, then $y=8+5=13$	5	13
if $x=$ 8, then $y=8+8=16$	8	16
if $x=10$, then $y=8+$	10	
if $x=11$, then $y=$	11	
etc.	14	
	19	

FIG. 97

3. Show that each pair of values of x and y in Fig. 97 satisfies equation (2) of problem 1.

4. Show that many other pairs of values of x and y can be found as those in Fig. 97 were found.

From 2, 3, and 4 it follows that we cannot tell from equation (2), *alone*, which pair of values in Fig. 97 gives the weights that A and B lift.

If B lifts 9 times as much as A, then the pair 9 and 1 (Fig. 97) satisfies problem 1.

x	y
4	20
5	19
8	16
7	
2	
1	

FIG. 98

If B lifts 5 times as much as A, then the pair 10 and 2 satisfies problem 1.

To find the weights we must, therefore, know a second relation between them.

5. From Fig. 96, p. 97, show that

$$x+y=24. \qquad (3)$$

Make a table (Fig. 98) of ten pairs of values of x and y, that satisfy equation (3). For convenience, solve equation (3) for y, thus

$$y=24-x. \quad \text{What axiom?} \qquad (4)$$

6. Which pair of numbers in Fig. 98 is also in Fig. 97? Show that these numbers satisfy equation (3), problem 5, and equation (2), problem 1, and that they are the numbers of pounds that A and B lift. Does any other pair of numbers in either table satisfy both equations?

7. Make a table of pairs of values of R and S that satisfy the equation

$$R-2S=8. \qquad (1)$$

For convenience, solve equation (1) for R, thus

$$R=8+2S. \qquad (2)$$

8. Make a table of pairs of values of R and S that satisfy the equation

$$R+S=14. \qquad (3)$$

9. Observe that many pairs of values of R and S can be found that satisfy equation (1), *alone*, and equation (3), *alone*. Find a pair that satisfies both equations.

10. Find as in problems 7, 8, and 9 the pair of values of H and K that satisfies both of the equations:

$$H+2K=20$$
$$-H+3K=-15.$$

11. Find the pair of values of M and L that satisfies the equations

$$M-2L=3$$
$$M+2L=11.$$

12. A weight of 84 lb., hanging on a stick 8 ft. long at a point 2 ft. from the middle, is raised by two boys who lift at the ends of the stick. How much does each boy lift?

74. From the foregoing problems it follows (1) that an equation containing two unknowns is satisfied by many pairs of numbers, (2) that two different equations in the same two unknowns may be satisfied by a single pair of numbers.

75. The single pair of values that satisfies the conditions of some problems can be found from two equations in two unknowns *directly* as follows.

1. The sum of two numbers is $24\frac{1}{2}$, and the difference is $8\frac{1}{2}$. What are the numbers?

Let s represent the smaller number, and l the larger. Then by the conditions of the problem

	$l+s=24\frac{1}{2}$	(1)
	$l-s=8\frac{1}{2}.$	(2)
Adding (1) and (2)	$2l=33$	(3)
	$l=16\frac{1}{2}.$	(4)
Subtracting (2) from (1)	$2s=16$	(5)
	$s=8.$	(6)

CHECK: The sum of $16\frac{1}{2}$ and 8 is 24; the difference (8 taken from $16\frac{1}{2}$) is $8\frac{1}{2}$.

Therefore the smaller number is 8 and the larger, $16\frac{1}{2}$.

2. The sum of two numbers is $24\frac{1}{4}$, and the difference is $11\frac{1}{4}$. What are the numbers?

3. The sum of two numbers is 12. If 3 times the smaller is subtracted from 4 times the larger, the result is 13. Find the numbers.

Letting l represent the larger number, and s the smaller,

$$l+s=12 \qquad (1)$$
$$4l-3s=13. \qquad (2)$$

To find l, multiply both sides of equation (1) by 3, and add the resulting equation to equation (2).

To find s, multiply both sides of equation (1) by 4, and subtract equation (2) from the resulting equation.

CHECK.

4. Solve for w and t:

$$6w+4t=42$$
$$5w-3t=16.$$

76. To solve beam or lever problems in two unknowns it is sometimes convenient to use, besides the law of leverages used thus far, another law which is explained in the following experiments.

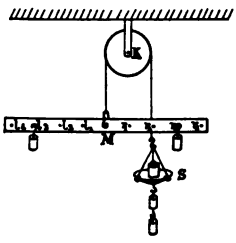


FIG. 99

1. Putting a weight, x, at l_3 (Fig. 99), and an equal weight, x, at r_3, it will be found that two weights, each equal to x, hung to the pan, S, will hold the apparatus in balance. Beginning on the left, record the relations for balance thus:

$$-x+2x-x=0.$$

2. With two weights, each equal to x at l_3, and two weights, each equal to x, at r_3, 4 weights, each equal to x, must be put at S for balance. But 3 or 5 weights, x, at S will be found not to balance. Make the record and interpretation thus:

Record	*Interpretation*
$-2x+4x-2x=0$	Balance
$-2x+3x-2x=-x$	Movement downward
$-2x+5x-2x=+x$	Movement upward.

3. If 2 weights, x, be put at l_3, 1 weight, x, at l_2, 1 weight, x, at r_2, and 2 weights, x, at r_3, 6 weights, each equal to x, at S will balance the apparatus, but neither 5 nor 7 weights, x, will balance it. The results will run thus:

Record	*Interpretation*
$-2x-x+6x-x-2x=0$	Balance
$-2x-x+5x-2x-x=-x$	Bar moves downward
$-2x-x+7x-x-2x=+x$	Bar moves upward.

4. Write the equations and state the results as shown by a beam for each loading of the following table:

No.	l_4	l_3	l_2	l_1	S	r_1	r_2	r_3	r_4
I	x	x	x	x	$8x$	x	x	x	x
II	x	x	0	0	$3x$	0	0	x	x
III	y	0	0	y	$6y$	y	0	0	y
IV	$2y$	0	$3y$	0	$15y$	$8y$	0	$2y$	0
V	$2y$	y	$3y$	y	$18y$	$9y$	0	$3y$	0

5. Use the same bar, supported as shown in Fig. 100, and balance it. If a weight, x, is placed at both L and R, two weights, each equal to x at M balance the bar. The results are:

Record	*Interpretation*
$+x-2x+x=0$	Balance.

Try the same weights at L and R and $3x$ at M; x at M.

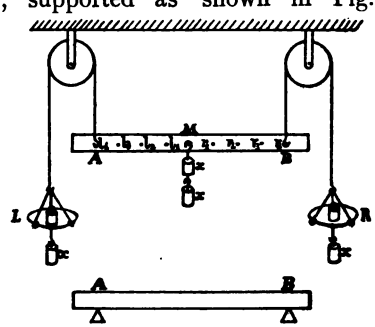


FIG. 100

6. Eight weights, x, at l_1, 3 weights, x, at R, and 5 at L, will balance the bar. If the 8 weights, x, are at r_1, then 3 weights, x, at L and 5 weights, x, at R are needed for balance. The results are:

Record	*Interpretation*
$+5x-8x+3x=0$	?
$+3x-8x+5x=0$	?

7. Write the equations and state whether or not there is balance for these experiments:

No.	L	l_4	l_3	l_2	l_1	M	r_1	r_2	r_3	r_4	R
I	$6x$	0	0	$8x$	0	0	0	0	0	0	$2x$
II	$8x$	0	0	$8x$	0	$4x$	0	0	0	0	$4x$
III	$8x$	x	0	$6x$	0	$4x$	0	0	0	x	$5x$
IV	$9x$	x	0	$8x$	0	$4x$	0	0	0	x	$5x$

8. When a bar is supported in two places (A and B, Fig. 100), it is called a *beam*. In the preceding experiments what is the test as to whether the bar, or beam, balances? Observe that in all these experiments the forces are parallel to each other.

77. The two equations used in the following problems are derived from the following laws:

Law of Forces.—*The algebraic sum of all the forces acting upon the material object (bar or beam) must equal zero, for balance.*

Law of Turning-Tendencies or Leverages.—*For balance, the algebraic sum of all the turning-tendencies must equal zero.*

Problems Applying Two Unknowns

1. A box 12 ft. long of a three-horse coal wagon is loaded with 6 tons of coal. If the box extends 2 ft. in front of the front axle and 4 ft. back of the rear axle, what are the weights on the front and rear axles? See Fig. 101.

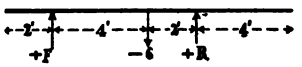


FIG. 101

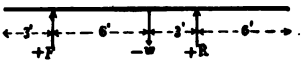


FIG. 102

2. A lumber wagon is coupled out to a distance of 9 ft. between the axles and loaded with a pile of lumber $3\frac{1}{2}$ ft. $\times 4$ ft. $\times 18$ ft. The load extends 3 ft. in front of the front axle and the material averages 48 lb. per cubic foot. What are the pressures on the axles due to this load? Fig. 102.

3. A wagon box, EFGO (Fig. 103), 10 ft. long and loaded with 40 bu. of wheat weighing 60 lb. per bu., extends $1\frac{1}{2}$ ft. in front of the front axle, B, and $2\frac{1}{2}$ ft. behind the rear axle, A. What is the load on each axle?

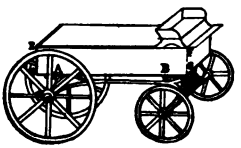


FIG. 103

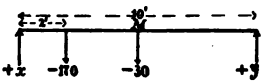


FIG. 104

4. Suppose a bar 10 ft. long, weighing 30 lb., is used by two men, one grasping it at each end, to carry a load of 170 pounds. How many pounds must each man carry, if the load is attached 2 ft. from the left end?

Measuring lever-arms from the middle point of the bar show that the equations are

$$x+y=200 \qquad (1)$$

$$-5x+510-0+5y=0. \qquad (2)$$

The zero term in (2) arises from the leverage of the weight of the bar, which is 30·0, zero being the lever-arm. But 30·0=0, for mani-

festly, a weight hanging at the middle point can have no tendency to turn the bar around this point, or, what amounts to the same thing, the turning-tendency about this point equals zero.

Solve the equations.

5. A foot-bridge 15 ft. long between supports (15 ft. span) rests on timbers at L and R (Fig. 105). The bridge weighs

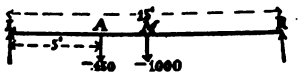


FIG. 105

1,000 pounds. Two men, whose combined weight is 450 lb., stand just over A, 5 ft. from the end L. Find the pressures on the supports at L and R, using M as turning-point.

6. A bridge 20 ft. long weighs 2,400 lb. and supports two loads; one of 600 lb., 4 ft. from the left end, and the other, 800 lb., 15 ft. from the left. What are the loads borne by the supports?

7. If, with the bridge of problem 6, four loads of 450 lb. each are placed, one 2 ft. from the left support, the second 6 ft., the third 9 ft., and the fourth 16 ft. from the left end, what are the upward forces exerted against the ends of the bridge by the supports?

8. A wagon standing on a culvert, A B (Fig. 106), has on the front axle a load of 2,500 lb., and on the rear axle 3,000

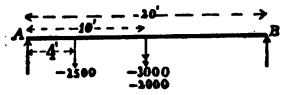


FIG. 106

pounds. The front wheels are 4 ft. and the rear wheels 10 ft. from the left end of the culvert. If the culvert is 20 ft. long, and weighs 2,000 lb., what are the pressures at the supports?

9. What would the pressures be if the front wheels stood 9 ft. from the left support (wagon coupled to 6 ft. between the axles)?

10. Two men, lifting at the ends of a stick 8 ft. long, raise a certain weight. What is the weight, and at what point does it hang, if one man lifts 25 lb., and the other 75 pounds? (Use M as turning-point, Fig. 107.)

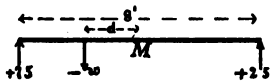


FIG. 107

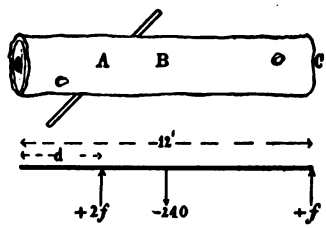


FIG. 108

11. Three boys desire to carry a 12-ft. log, weighing 240 pounds (Fig. 108). Two of the boys lift at the ends of a hand-spike placed crosswise underneath the log and the third boy carries the rear end of the log. Where must the hand-spike be placed that all may lift equally? (Use O as turning-point.)

12. Solve problem 11 using A as turning-point. Observe that the distance AB is $6-d$.

EXERCISE XIV

Find the values of the unknown numbers, and check:

1. $s-14+t=0$
   $s-t+2=0$

2. $(4)(w)+(-6)(+3)+(-f)(+1)=0$
   $w-11+3f=0$

3. $(2\frac{1}{2})(3k)+(16)(-3)+(2)(3l)=0$
   $-2k-10+6l=0$

4. $-3(10-w)+(-2)(+91)+4x=0$
   $-w-46+x=0$

5. $w+16-3t=0$
   $5w-2t-11=0$

6. $-3\frac{1}{2}x-14+3\frac{1}{2}y=0$
   $x+y=12$

7. $\frac{1}{2}l+\frac{1}{6}r+1=0$
$\frac{1}{3}l+7-\frac{3}{4}r=0$

8. $\frac{2(x+5)}{3}-\frac{5y}{2}=-31$

$\frac{3}{8}x+\frac{3(2y-7)}{4}=21\frac{3}{4}.$

SUMMARY

1. Beam problems in which one number is unknown may be solved by using the following law:

Law of Turning-Tendencies. *For balance, the algebraic sum of all the turning-tendencies must equal zero.*

2. Beam problems in which two numbers are unknown may be solved by using the law of turning-tendencies, and the following law:

Law of Forces. *The algebraic sum of all the forces must equal zero, for balance.*

When only one number is unknown the Law of Forces may be used alone.

3. An equation containing two unknowns is satisfied by many pairs of numbers.

Two different equations in the same two unknowns may be satisfied by a single pair of numbers.

4. To solve a problem leading to two equations in two unknowns, a third equation is derived which contains only one of the unknowns.

5. The product of two factors is zero, if one of the factors is zero.

CHAPTER VI

PROBLEMS IN PROPORTION AND SIMILARITY

Drawing to Scale

78. Problems in finding distances may be solved by drawings made to scale.

1. A man starting at O, Fig. 109, walks 45 yd. east and then 60 yd. north. What is the direct distance from the

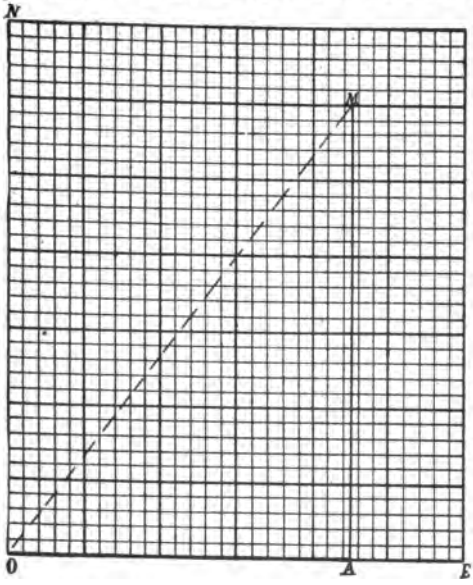


FIG. 109

stopping-point to the starting-point, if 1 cm. on the drawing represents 10 yards?

2. Find the direct distance from the starting-point to stopping-point in problem 1 from a drawing in which 1 cm. represents 15 yards.

3. A man walks 80 yd. south, then 144 yd. east, and then 120 yd. north. Find, by a diagram, his distance from the starting-point, letting 1 cm. on squared paper represent 12 yards.

4. Two men start from the same point. One walks 5 mi. west, and then 3 mi. north; the other walks 4 mi. south, and then 5 mi. east. How far apart are they?

5. Draw a line 3 in. long, and let it represent a distance of 48 feet. What distance is represented by 1 inch? By 2 inches? By 6 inches? By $1\frac{1}{2}$ inches? By $2\frac{3}{8}$ inches? By $1\frac{1}{16}$ inches?

In problem 5, the drawing is said to be *made to a scale* of 1 inch to 16 feet.

6. Draw to the same scale: 8 feet; 12 feet; 24 feet; 28 feet.

7. If a line 5 in. long represents a distance of 75 mi., what is the scale?

8. If .7 of an inch on a map represents a distance of 21 mi. on the earth, how many inches represent 87 miles? What scale is used?

9. Draw a plan of a rectangular field 16 rods long and 12 rods wide, using the scale of 1 in. to 4 rods (1 in. = 4 rods), and find the distance in rods, diagonally across the field.

Indicate the scale on all scale-drawings.

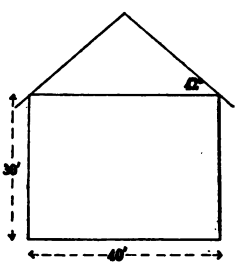


FIG. 110

10. Draw to the scale, $\frac{1}{2}$ cm. to 1 ft., a plan of a room 24 ft. by 18 ft., and find the distance diagonally across the floor.

11. Draw to the scale, 2 cm. to 5 ft., a plan of the end of the house in Fig. 110, and find the height of the top of the roof from the ground. Use a protractor to draw the angle 42°.

Ratio

79. The ratio of 6 to 3 is $\frac{6}{3}$, or 2; of 3 to 4 is $\frac{3}{4}$; of a to b is $\frac{a}{b}$. The ratio of 6 to 3 is sometimes written 6 : 3; of 3 to 4, 3 : 4; and of a to b, $a : b$.

80. The **ratio** of any number to another number is the *quotient* found by dividing the first number by the second. Thus $\frac{2}{3}$ is the ratio of 2 to 3. Any fraction may be regarded as an expression of the ratio of the numerator to the denominator.

1. Write in two other ways the following ratios:

(1) 5 to 20	(7) $a+b$ to c
(2) 18 to 25	(8) $x+y$ to $c+d$
(3) 25 to 18	(9) $a-b$ to $c+d$
(4) x to y	(10) $ax+ay$ to a
(5) c to d	(11) $3x+2$ to ab
(6) d to c	(12) $a+b$ to x.

2. How do two numbers whose ratio is 1 compare in size?

3. What is the ratio of the cost of 5 yd. of silk at \$1.50 to that of 50 yd. of cotton at $12\frac{1}{2}$ cents?

4. What is the ratio of the length of the field (problem 9, p. 108) to the width? Of the length of the plan to the width?

5. What is the ratio of the length of the room (problem 10, p. 108) to the width? Of the length of the plan to the width?

6. What is the ratio of 1 yd. to 1 foot? Of 1 yd. to 1 inch? Of 1 yd. to 6 inches? Of 3 yd. to 3 inches? Of 3 yd. to 3 feet?

7. What is the ratio of 1 lb. to 1 ounce? Of 1 oz. to 5 pounds? Of 1 ton to 500 pounds? Of 5 lb. to 5 ounces?

8. What is the ratio of 1 mi. to 1 yard? Of 1 mi. to 1 foot? Of 1 mi. to 880 feet? Of 1 mi. to 880 miles?

81. Problems 6, 7, and 8 show that magnitudes must be expressed in the same unit before the ratio can be expressed as a single number.

1. Draw three triangles in each of which the angles are respectively 35°, 65°, and 80°. Are all the triangles necessarily of the same size? Do all have the same shape?

2. Draw three triangles of different sizes in each of which the angles are 30°, 60°, and 90°. Compare the triangles as to shape. Measure all sides of the triangles. Find the ratios of the sides of the first triangle to the corresponding sides of the second; to the corresponding sides of the third triangle.

3. Draw triangles of different sizes having the angles 52°, 112½°, and 15½°. Compare the triangles as to shape. Find the ratio of the sides of one of the triangles to the corresponding sides of another.

4. Draw three triangles each of which has angles 90°, 25°, and 65°. Are all of the triangles necessarily of the same size and shape? Compare the ratios of corresponding sides.

5. Draw a triangle. Draw another having the angles equal respectively to the angles of the first triangle. Are the two triangles necessarily of the same size and shape? Compare the ratios of the corresponding sides.

Similar Triangles

82. Triangles having the same shape are called **similar triangles**. Similar triangles are not necessarily of the same size.

1. Draw a triangle with sides 4 in. and 5 in., respectively, and an angle of 50° included between them. First draw it actual size and then to the scale of $\frac{3}{8}$ in. = 1 inch. Are the triangles similar? Why? Measure with a protractor the pairs of corresponding angles. Find the ratios of the corresponding sides.

2. Two sides of a triangle are $2\frac{3}{8}$ cm. and 5.5 centimeters. The included angle is 60°. Two sides of another triangle are $4\frac{3}{4}$ cm. and 11 cm., and the included angle is 60°. Draw the triangles. Are they similar? Give reason for the answer. Measure the pairs of corresponding angles and find the ratios of the corresponding sides.

3. Draw a triangle. Draw another with sides respectively double the lengths of the first. Compare the triangles as to shape. Measure the corresponding angles. What is the ratio of the corresponding sides?

4. Two sides of a triangle are 4.5 cm. and 9.5 cm.; the included angle is 70°. Two sides of another triangle are $5\frac{3}{8}$ cm. and $13\frac{1}{4}$ cm. and the included angle is 70°. Are the triangles similar? Compare the corresponding angles. Find the ratios of the corresponding sides.

83. The triangles of problems 1, 2, 3, **§82,** though differing in size, have the same shape and have the corresponding angles equal. Notice also that each side of the smaller triangle (problem 1) is $\frac{3}{8}$ of the length of the corresponding side of the larger triangle.

All *similar* triangles may be regarded as the same triangle drawn to different scales. They may be regarded as the same triangle *magnified*, or *minified* to a definite scale.

1. Draw two triangles having the same shape but different sizes. Measure two corresponding pairs of sides in each and compare the ratios. Measure another pair of corresponding sides and compare the ratios.

2. According to problems 1, 2, 3, **§82,** and problem 1, **§83,** what seems to be true of the ratios of corresponding pairs of sides of triangles having the same shape (similar triangles)?

84. *Two triangles are similar when the corresponding angles are equal and when the ratios of the corresponding sides are equal.*

1. In the two similar triangles of Fig. 111, (1), if $a=4$ in., $A=12$ in., and $b=4$ in., how long is B?

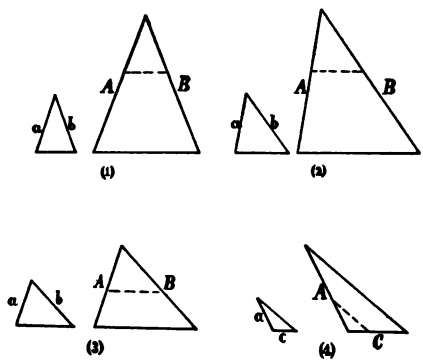


FIG. 111

2. In the similar triangles of Fig. 111, (2),

If $a=4$ in., $A=12$ in., and $b=5$ in., how long is B?
If $a=3$ in., $b=8$ in., and $B=32$ in., how long is A?
If $a=x$ in., $b=8$ in., and $B=32$ in., how long is A?

3. In the similar triangles in Fig. 111, (3),

If $A=21$ in., $b=9$ in., and $B=27$ in., how long is a?
If $a=5\frac{1}{2}$ in., $A=22$ in., and $B=30$ in., how long is b?

4. In the similar triangles in Fig. 111, (4),

If $a=3$ in., $A=8$ in., and $C=5$ in., how long is c?
If $A=24$ in., $c=4$ in., and $C=7$ in., how long is a?
If $A=y$ in., $c=4$ in., and $C=7$ in., how long is a?

5. In Fig. 112, if the stake 3 ft. high casts a shadow 8 ft.

long, and the tree, at the same time, casts a shadow 80 ft. long, how high is the tree?

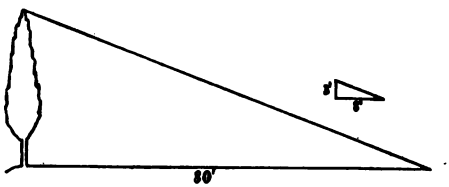


FIG. 112

6. Are triangles O A B and O H K (Fig. 113) similar? Give reason for answer. How long is x?

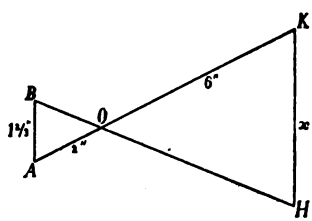


FIG. 113

7. A boy holds a pencil, A B (Fig. 114), 2 ft. from his eye, so that it covers a flag-pole 360 ft. distant. To make the triangles E A B and E F K similar, how must the pencil be held? If the pencil is 6 in. long, how high is the pole?

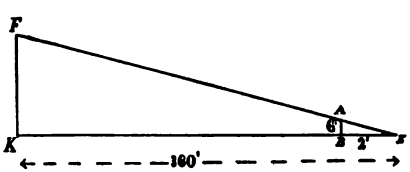


FIG. 114

8. A lumberman who is 5 ft. tall wishes to find a tree 60 ft. to the first limbs. He drives a stake in the ground and places

his feet against it as in Fig. 115. If the stake is 4 ft. high, how far must it be placed from the foot of the tree, that he may determine whether or not the trunk is 60 ft. to the limbs?

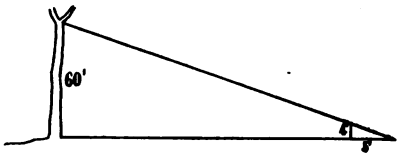


FIG. 115

9. The gables of a house and of a porch have the same shape. The sides of the porch-gable are 7 ft., 7 ft., and 10 feet. The longest side of the house-gable is 25 feet. What is the ratio of the corresponding sides? How long are the other two sides of the house-gable?

10. The sides of a triangle are 8, 10, and 13. The shortest side of a similar triangle is 11. What is the ratio of the corresponding sides? Find the other sides.

11. The sides of a triangle are 4.6 cm., 5.4 cm., and 6 centimeters. The corresponding sides of a similar triangle are x cm., y cm., and 15 centimeters. Find x and y.

12. The sides of a triangle are 1, 2, and 3, and the longest side of a similar triangle is 20. Find the other sides of the second triangle.

13. Draw a triangle (Fig. 116) having two of the sides A B and A C equal to 10 in. and 12 in., including any con-

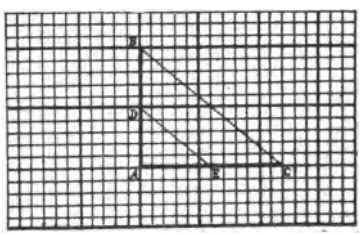


FIG. 116

venient angle between them. Call the third side the base. Through a point on A B, 5 in. from the vertex A, draw D E parallel to the base. Measure the distance A E. How does the ratio of the corresponding parts of the sides A B and A C, cut off by the parallel D E, compare with the ratio of the sides A B and A C?

14. Draw a parallel to the base (Fig. 116) through a point of the 10-in. side, $2\frac{1}{2}$ in. from the vertex, and measure the distance from the vertex to the crossing-point of the $2\frac{1}{2}$ in. parallel with the 12-in. side. Compare the ratio of the corresponding parts of the sides with the ratio of the sides themselves.

15. Compare the ratios of the corresponding parts of the sides made by parallels to the base, through a point of the 10 in.-side 3 in. from the vertex; 6 in. from the vertex; $1\frac{1}{4}$ in. from the vertex; $7\frac{1}{2}$ in. from the vertex.

85. *A line drawn parallel to one side of a triangle divides the other two sides into corresponding parts having the same ratio as the sides themselves.*

Any number of parallels to the base of a triangle divide the other two sides into parts having the same ratio.

1. In Fig. 117, A B = 21, A C = 35, and A D = 3. D E is parallel to B C. Find the value of x.

2. Show that triangles A D E and A B C, Fig. 116, are similar.

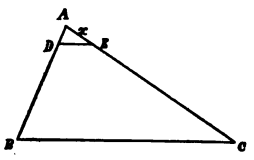


FIG. 117

86. *If a line is drawn parallel to one side of a given triangle, meeting the other two sides, a triangle is formed which is similar to the given triangle.*

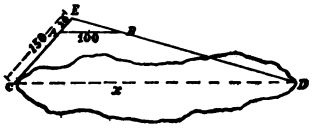


FIG. 118

1. A B (Fig. 118) is parallel to C D. Find C D, the distance across the lake.

2. Draw a triangle, as A B C, Fig. 119. Divide A B and A C into parts having the ratio 1:3. Connect the points of

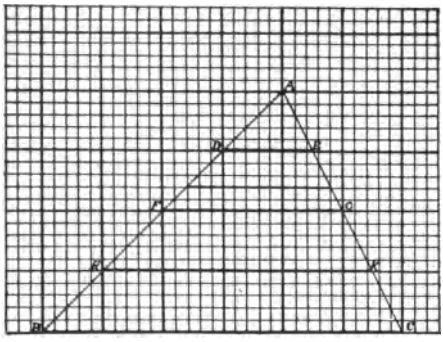


FIG. 119

division, D and E, by a straight line. Divide A B and A C in other ratios, as 2:3, 1:1, 3:7, 1:9, 1:4, and connect the corresponding points of division by straight lines. Notice that these lines are parallel, that is, they will not meet, however far extended.

87. *If two sides of a triangle are divided into parts having the same ratio, the line joining the points of division is parallel to the third side of the triangle.*

Problems in Surveying

88. Problems in finding directions and distances may be solved by drawings made to scale.

In Fig. 121 the direction shown by the arrow is read "30° east of north;" in Fig. 122, "50° west of south;" in Fig. 123, "20° east of south."

89. The direction of a line, when indicated by the angle it makes with the north-south line, is called the *bearing* of the line.

Surveyors use the *surveyors' compass*, Fig. 128, to measure the bearings of lines.

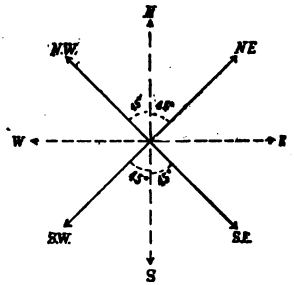


FIG. 120

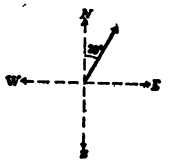


FIG. 121

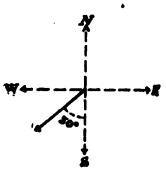


FIG. 122

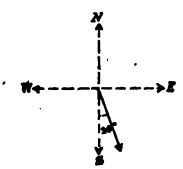


FIG. 123

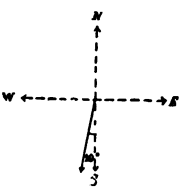


FIG. 124

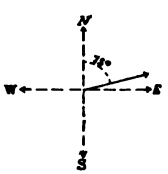


FIG. 125

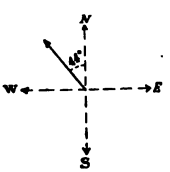


FIG. 126

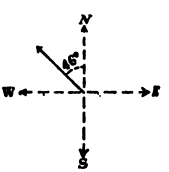


FIG. 127

1. Read the bearings of the arrows in Figs. 124, 125, 126, 127.

2. With a ruler and protractor draw lines having the following bearings:

65° east of south	47½° west of south
65° east of north	43° west of north.

90. The bearing of a point B from a point A is the bearing of the line A B with reference to the north-south line through A.

1. In Fig. 129, read the bearings of

A from O	B from O	C from O
O from A	O from B	O from C.

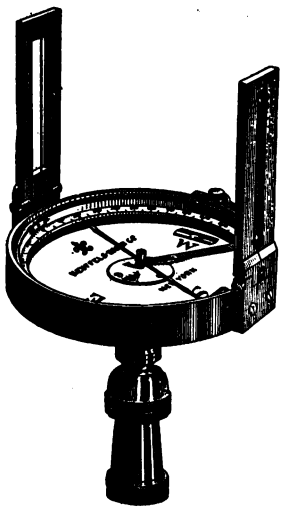


FIG. 128

2. The bearing of fort A from fort B, both on the sea-coast, is 65° west of north. An enemy's vessel at anchor off the coast is observed at A to bear northeast; at B, northwest. The forts are known

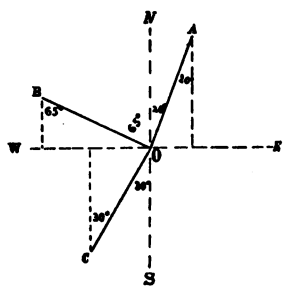


FIG. 129

to be 7 mi. apart. Find by drawing a plan (scale: 2 cm. = 1 mi.) the distance from each fort to the vessel.

In solving problems like 2, draw first a sketch of the distances and directions, and then make the scale drawing.

3. Find the distance P Q, if Q is 6.4 mi. east and 9.8 mi. north of P (scale: 1 in. = 2 mi.). What angle does P Q make with the north-south line through P?

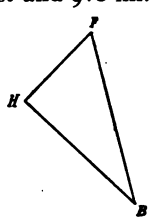


FIG. 130

4. A hill in a battle-field obstructs the view from a battery at B, Fig. 130, to the enemy's fort at F. A point H is found at the bottom of the hill, from which F is observed to bear

4 mi. northeast. If H is 6.25 mi. northwest of B, what is the distance F B, and the bearing of F from B?

The triangle in the drawing, problem 4, is *similar* to the triangle in the field, for the first triangle is the same as the second with all sides *minified* or *reduced* in a definite ratio, the angles remaining unchanged.

5. A tree at Q is 6.5 rods north of R, and 9 rods west of S. What is the distance and bearing of S from R? Show that the triangle in the drawing is *similar* to the triangle in the field.

6. A man wishes to measure the width of a river without crossing it. The river flows due west. Standing at A, on

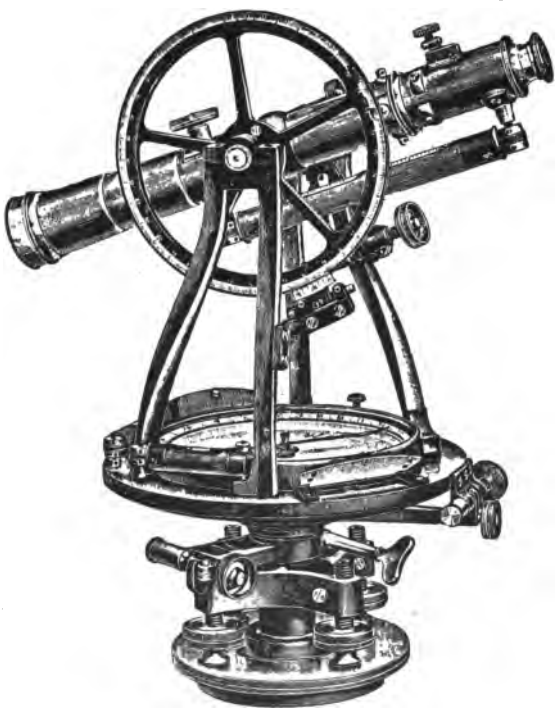


FIG. 131

the bank, he observes a tree on the other bank in the direction, 20° east of north. He walks 50 rods east along the bank to B, and there observes the tree in the direction, 60° west of north. Find the width of the river.

91. In the preceding problems angles have been measured by referring them to the north-south line. By means of the *engineers' transit* (Fig. 131) angles in any position may be measured.

For rough measurements, an angle-measurer can be constructed by tacking a protractor on a board (Fig. 132). A ruler with a pin stuck in it at each end can be used for sighting.

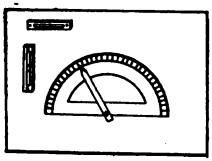


FIG. 132

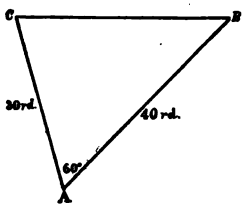


FIG. 133

1. Draw a plan of a garden plot from the data of Fig. 133. Find the length, in rods, of B C, and of the perpendicular from B to A C. (Scale: 1 in. = 10 rods.)

2. Draw triangles having the following parts:

4 in., 4 in., and the included angle 60°
3 in., 4 in., and the included angle 90°
3 in., $3\frac{1}{2}$ in., and the included angle 50°.

Measure the third sides of the triangles. Compare your results with those of other members of the class. What truth about triangles do you infer?

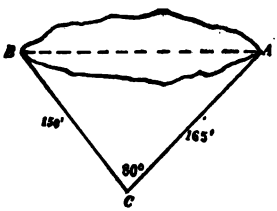


FIG. 134

3. A railroad surveyor wishes to measure across the swamp A B, Fig. 134. He measures the distance from a tree at A to a stone at C and finds

it to be 165 feet. The distance from a tree at B to the stone is 150 feet. Find the distance in feet across the swamp, the angle at C being 80° (scale: 1 cm. = 15 ft.). Show that the triangle in the drawing is similar to the surveyed triangle.

4. To measure the width, A C, of a stream (Fig. 135), without crossing it, an engineer lays off a line, B C, on one side of the river, and measures (with a transit) the angles at B and C. Draw a triangle to scale from the data in the figure, and determine the width of the river.

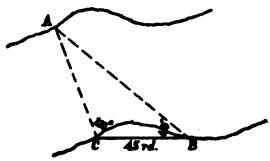


FIG. 135

5. Draw triangles from the following data:

A B = $3\frac{1}{2}$ in., angle A = 15°, angle B = 130°
A B = $3\frac{1}{2}$ in., angle B = 90°, angle C = 60°
A B = $3\frac{1}{2}$ in., angle C = 55°, angle A = 20°.

In each triangle measure the sides not given and compare as in problem 2. What do you infer?

6. A boy wishes to determine the height, H K (Fig. 136), of a factory chimney. He places the angle-measurer first at B and then at A and measures the angles x and y. The

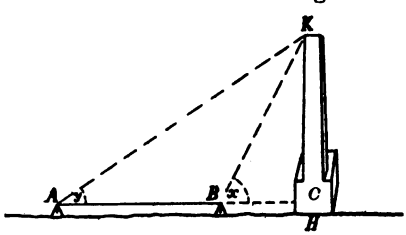


FIG. 136

angle-measurer lies on a box, or tripod, $3\frac{1}{2}$ ft. from the ground. A and B are two points in line with the chimney and 50 ft. apart. What is the height of the chimney if the ground is level and if $x = 63°$ and $y = 33\frac{1}{2}°$?

7. Classify triangle ACK, Fig. 136, as to its angles. Triangle BCK.

8. How many degrees are there in each angle at K, Fig. 136? In the angle adjacent to angle x?

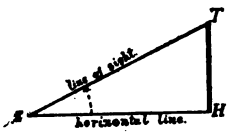


FIG. 137

92. A telescope is pointed horizontally toward a tower (Fig. 137), and the farther end is then raised (elevated) until the telescope points to the top of the tower. The angle through which the telescope turned is the **angle of elevation** of the top of the tower, from the point of observation.

1. From point A, Fig. 136, what is the angle of elevation of the top of the chimney? From point B?

2. When the angle of elevation of the sun is 25°, a building casts a shadow 90 ft. long, on level ground. Find the height of the building.

3. Find the angle of elevation of the sun when a tree 40 ft. high casts, on level ground, a shadow 60 ft. long.

4. On the top of a tower stands a flagstaff. At a point, A, on level ground, 50 ft. from the base of the tower, the angle of elevation of the top of the flagstaff is 35°. At the same point A, the angle of elevation of the top of the tower is 20°. Find the length of the flagstaff.

93. A telescope at T, on the top of a cliff (Fig. 138), is pointed horizontally, and then the farther end is lowered (depressed) until the telescope points to the buoy at B. The angle through which the telescope turned is the **angle of depression** of the buoy from the point T.

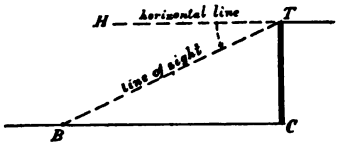


FIG. 138

1. If the height of the cliff, Fig. 138, is 100 ft., and the angle of depression of the buoy, as seen from T, is 40°, what is the distance of the buoy from the bottom of the cliff?

2. A boat passes a tower on which is a search-light 120 ft. above sea-level. Find the angle through which the beam of light must be depressed, from the horizontal, so that it may shine directly on the boat when it is 400 ft. from the base of the tower.

3. From the top of a cliff 150 ft. high, the angle of depression of a boat is 25°. How far is the boat from the top of the cliff?

4. From a lighthouse, situated on a rock, the angle of depression of a ship is 12°, and from the top of the rock, it is 8°. The height of the lighthouse above the rock is 45 feet. Find the distance of the ship from the rock.

Proportion

94. Comparing the ratios of the areas of figures with the ratios of corresponding dimensions leads to proportion.

1. Two rectangles, as ABCD and EFGH (Fig. 139), have equal altitudes, *h*. The bases are 7 in. and 4 in. respectively. What are the areas? Find the ratio of the areas. Find the ratio of the bases. How does the ratio of the areas compare with the ratio of the bases?

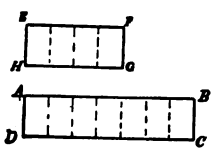


FIG. 139

2. The altitude of a rectangle is 10 in. and the base is 4 feet. The altitude of another rectangle is 20 in. and the base is 4 feet. What is the ratio of the areas? Of the altitudes? Compare the ratios.

3. Two rectangles have bases 20 ft. and 25 ft., and an altitude of 15 feet. Express by an equation that the ratio of the areas equals the ratio of the bases.

4. The dimensions of one rectangle are a and b, and of another a and c. Compare the ratio of the areas with the ratio of the unequal dimensions, and express the result by an equation.

5. How does the ratio of the areas of rectangles having equal bases compare with the ratio of the altitudes? How does the ratio of areas of rectangles having equal altitudes compare with the ratio of the bases?

6. The area of rectangle ABCD (Fig. 140), is 80 sq. ft., and the base is 10 yards. What is the area x of rectangle EFGH, having the same altitude and a base equal to 24 yards?

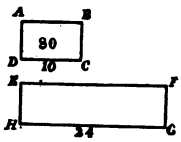


FIG. 140

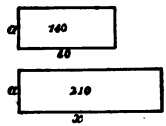


FIG. 141

7. Express by an equation the relation between the areas and bases of the rectangles in Fig. 141. Find the base x.

8. The equation $\frac{5}{4}=\frac{x}{8}$ expresses the relation between the ratio of the areas and the ratio of the altitudes of two rectangles. Find the altitude x.

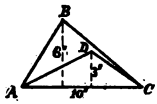


FIG. 142

9. Triangles ABC and ADC (Fig. 142), with the same base, have altitudes as shown. What is the ratio of the areas? Of the altitudes? How do the ratios compare?

10. If two or more triangles have equal bases, how does the ratio of the areas of any two compare with the ratio of the altitudes?

FIG. 143

11. Two triangles (Fig. 143) have equal bases, b, and altitudes as shown. The ratio of the areas is $\frac{3}{2}$. Express by an equation the

relation between the ratio of the areas and the ratio of the altitudes. Find the altitude, x, by solving the equation.

12. Compare the ratios of areas and of bases of triangles with equal altitudes (Fig. 144). Express the result by an equation.

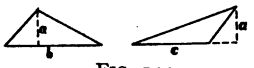


FIG. 144

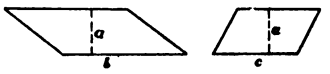


FIG. 145

13. Compare the ratios of areas and of bases of the parallelograms in Fig. 145. Express the result by an equation.

95. An equation of two ratios is called a **proportion.** For example, $\frac{4}{6}=\frac{2}{3}$, $\frac{a}{b}=\frac{ac}{bc}$, and $\frac{a}{b}=\frac{c}{d}$ are all called proportions, and are sometimes written thus: $4:6=2:3$, $a:b=ac:bc$, and $a:b=c:d$. The last may be read "a is to b as c is to d." Read the other two. Numbers that form a proportion are said to be *proportional.*

Four lines are said to be proportional if their lengths are proportional.

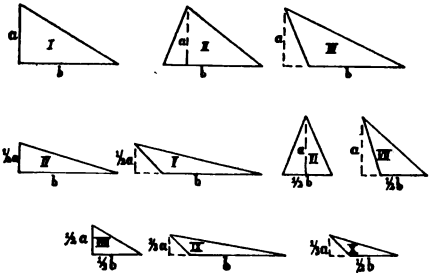


FIG. 146

1. In Fig. 146, the letters a and b denote the same numbers throughout. How do the areas of triangles I, II, and III compare? I and IV? II and VI? III and V? IV and VIII? II and IX? IX and X? VII and X? III and X?

2. Show that (1) areas of rectangles are proportional to the bases, if the altitudes are equal, i. e., the ratio of the areas equals the ratio of the bases;

(2) areas of rectangles are proportional to the altitudes, if the bases are equal;

(3) areas of rectangles are proportional to the products of the bases and altitudes;

(4) areas of triangles are proportional to the products of the bases and altitudes.

3. The altitude and base of a triangle are 4 ft. and 15 ft. respectively. What are the dimensions of other triangles of $\frac{1}{2}$ the area? Of $\frac{1}{3}$ the area? Of $\frac{1}{6}$ the area?

4. The altitude and base of a rectangle are 6 in. and 8 in. respectively. What are the altitudes and bases of triangles whose areas are:

(1) equal to the area of the rectangle?
(2) twice the area of the rectangle?
(3) one-half the area of the rectangle?

5. Answer the same questions when the altitude and base of the rectangle are a and b inches respectively.

6. A triangle and a rectangle have equal bases and are equal in area. How do the altitudes compare?

7. The dimensions of a rectangular block are 4 ft. $\times$ 15 ft. $\times$ 25 feet. What are the dimensions of other rectangular blocks having $\frac{1}{2}$ the volume? $\frac{1}{3}$ the volume? $\frac{1}{5}$ the volume? $\frac{1}{6}$ the volume? $\frac{1}{10}$ the volume?

8. Two rectangular flower beds have the same shape, but are different in size. One is 3 ft. wide and 5 ft. long; the other is 12 ft. wide. How long is it? What is the ratio of the corresponding sides?

9. Two books have the same shape. One is $5\frac{1}{2}$ in. wide and $7\frac{1}{2}$ in. long. The other is 15 in. long. How wide is it?

10. The top of a desk and a rectangular sheet of paper, 12 in. by 18 in., have the same shape. The desk is 2 ft. wide. How long is it?

11. A city block and a lot within the block have the same shape. The lot is 100 ft. by 150 ft. and the block is 300 ft. wide. How long is it?

12. Is $2:5=5:15$ a proportion? Is $2:7=8:25$ a proportion? Give reasons for your answers.

96. The first and last terms of a proportion are called the **extremes**; the second and third, the **means.**

1. Compare the product of the extremes with the product of the means in the proportion $2:5=6:15$; in $3:7=6:14$; in $20:2=10:1$; in $12:3=4:1$. What do you find true of the products?

2. For which of the following expressions does the product of the first and last number equal the product of the other two?

(1) $1:3=4:12$	(6) $2:3=2a:3a$
(2) $3:4=6:12$	(7) $8:8a=3:3a$
(3) $2:3=8:11$	(8) $x:y=4x:4y$
(4) $5:6=10:12$	(9) $3a:3x=6a:6x$
(5) $8:3=15:3$	(10) $x:3x=1:3$.

Is there any proportion in this list in which the product of the means does not equal the product of the extremes?

Is there in the list any expression that is not a proportion, for which the product of the first and fourth number is equal to the product of the second and third?

97. *In a proportion, the product of the means equals the product of the extremes.*

This is a convenient test of proportionality.

1. By this test tell what expressions in problem 2, **§96,** are proportions.

2. If $\frac{a}{b}=\frac{c}{d}$, prove that $ad=bc$.

3. Divide 85 into two parts in the ratio 2:3.

4. Divide 84 into three parts proportional to 3:4:5.

5. What number added to 12 and subtracted from 30, gives results that are to each other as 5:10?

6. Two numbers are in the ratio of 5:6. If 12 is subtracted from each, the differences are in the ratio 3:4. What are the numbers?

7. The ratio of two lines is $2\frac{1}{2}:3\frac{3}{4}$. The longer line is 30 centimeters. Find the shorter line.

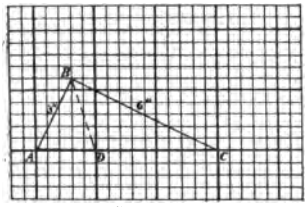


FIG. 147

8. In Fig. 147 the ratio of the parts of AC equals the ratio of AB to BC. AD is 2 in. less than DC. Find the lengths of AD and DC.

9. On squared paper make an accurate drawing for problem 8. Measure the angles at B. How do they compare in size?

10. If 6° be taken from one of two complementary angles and added to the other, the ratio of the two angles, thus formed, is 2:7. Find the angles.

11. The ratio of 2 times one of two supplementary angles to 8 times the other is 1:2. Find the supplementary angles.

12. Three angles just covering the plane around a point are to each other as 2:3:4. Find them.

13. The angles of a triangle are as 1:2:3. Find them.

14. The acute angles of a right triangle are as 2:5. Find them.

15. Find the value of x in the following proportions:

(1) $\frac{x}{3}=\frac{10}{30}$

(2) $\frac{4}{x}=\frac{132}{3}$

(3) $\frac{18}{810}=x:3$

(4) $\frac{x}{51}=\frac{10}{1020}$

(5) $\frac{5-x}{x+3}=\frac{2-x}{4+x}$

(6) $\frac{x-12}{1-3x}=\frac{2}{1}$

(7) $\frac{x+1}{10}=\frac{3x+2}{28}$

(8) $\frac{x-4}{x-13}=\frac{x-8}{x-14}$

(9) $\frac{x+2}{x+3}=\frac{x+3}{x+1}$

(10) $x^2:64=400:1$

(11) $\frac{x^3}{3}=\frac{27}{192}$

(12) $\frac{x+5}{4}=\frac{16}{x+5}$.

98. Proportions may be written from equations that express the equality of products.

1. The statements below are different arrangements of the four factors in the equation

$$8 \cdot 7=14 \cdot 4. \qquad (1)$$

Apply the test of proportionality and point out which statements are proportions:

(2) $8:14=4:7$

(3) $\frac{7}{4}=\frac{14}{8}$

(4) $4:7=8:14$

(5) $\frac{14}{8}=\frac{7}{4}$

(6) $8:7=14:4$

(7) $\frac{7}{14}=\frac{8}{4}$

(8) $8:4=7:14$

(9) $\frac{8}{14}=\frac{7}{4}$.

2. From what place in the given equation (1) were the *means* taken in the proportions? The extremes?

3. Is the same thing true of any of the statements that are not proportions?

4. Try to write a false expression of proportion by making both factors of either side of equation (1) the means of the expression.

5. Try to write a correct statement of proportion using any one of the four factors of the given equation for a mean and the other factor on the same side for an extreme.

6. Write four proportions from $3 \cdot 28 = 4 \cdot 21$, and apply the test of proportionality.

7. Write four proportions from $a \cdot 12b = 3a \cdot 4b$, and test.

99. *If the product of two numbers equals the product of two other numbers, either pair may be made the means and the other pair the extremes of a proportion.*

1. From each of the following equations write at least four proportions and show that they satisfy the test of proportionality:

(1) $3c \cdot 10d = c \cdot 30d$
(2) $6 \cdot 21a = 18 \cdot 7a$
(3) $15 \cdot 7r = 105 \cdot r$
(4) $a \cdot bc = ab \cdot c$
(5) $x \cdot \frac{4}{5} = \frac{2x}{5} \cdot 2$
(6) $a \cdot b = ab \cdot 1$.

100. Proportions may be written from other proportions.

1. Using the numbers of the proportion $4:7 = 12:21$, write another expression in proportion form:

(1) without changing the positions of the 4 and the 21 (the extremes),

(2) without changing the positions of the 7 and the 12 (the means).

2. Interchange the means only in $5:7 = 15:21$, and test for proportionality. Interchange the extremes and test.

3. If $a:b = c:d$, is $a:c = b:d$? Reason for answer.

4. If $a:b = c:d$, is $d:b = c:a$? Reason for answer.

101. If the means or the extremes of a proportion are interchanged the resulting expression is a proportion.

When a second proportion is made from a given proportion by interchanging the means, or by interchanging the

extremes, the second proportion is said to be obtained from the given proportion by *alternation*.

1. In Fig. 148, $\frac{AB}{AE}=\frac{AC}{AD}$.

Show that $\frac{AB}{AC}=\frac{AE}{AD}$.

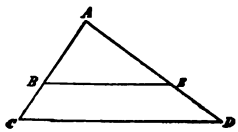


FIG. 148

2. In two equilateral polygons having the same number of sides, the corresponding sides are proportional.

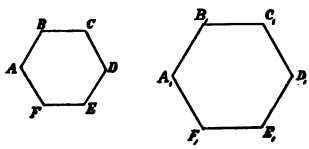


FIG. 149

PROOF: Since $\frac{AB}{BC}=1$ (Fig. 149), and $\frac{A_1B_1}{B_1C_1}=1$ (Why?), it follows that

$$\frac{AB}{BC}=\frac{A_1B_1}{B_1C_1}. \quad \text{Why?}$$

Then

$$\frac{AB}{A_1B_1}=\frac{BC}{B_1C_1}. \quad \text{Why?}$$

In a similar way prove

$$\frac{BC}{B_1C_1}=\frac{CD}{C_1D_1}=\frac{DE}{D_1E_1}=\cdots\cdots$$

3. If two parallel lines are intersected by three or more parallel lines, the ratio of any two parts of one of the first two parallels equals the ratio of the corresponding parts of the other.

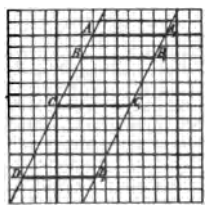


FIG. 150

Suggestion for the proof:

$$\frac{AB}{A_1B_1}=\frac{BC}{B_1C_1}. \quad \text{Why?}$$

Variation

1. Let the vertical side of a square (Fig. 151) represent 1 mile and the horizontal side 1 hour. Graph the distance passed over in 1 hour, 2 hours, 3 hours,..... 10 hours, by a man walking at the rate of 2 miles an hour.

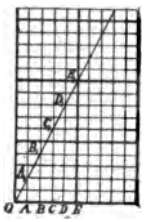


FIG. 151

Draw the lines $O A_1$, A_1B_1, B_1C_1..... What kind of a line is $O E_1$?

How do the angles of triangles $O A A_1$, $O B B_1$, $O C C_1$, etc., compare in size?

What is the ratio of $A A_1$ to $O A$? of $B B_1$ to $O B$? of $C C_1$ to $O C$? .....

2. Show that the triangles $O A A_1$, $O B B_1$, ..., etc., are similar.

What is the ratio of any distance-line to the corresponding time-line?

How does the distance change when the time is doubled, trebled, quadrupled, etc.?

How does the distance-line change (vary) when the time-line changes?

3. Write the equation for the time t and the distance d passed over by a man walking at the rate of 2 miles an hour. What are the values of d as t takes the values 1, 2, ... 10? What is the ratio of a value of d to the corresponding value of t?

4. On squared paper let the vertical side of a small square denote 1 cent and the horizontal side 1 pound. Graph the cost of flour at $3\frac{1}{2}$ cents per pound, for 1 lb., 2 lb., 3 lb.,..... 8 pounds.

How does the graph show the cost to change (vary) as the weight is doubled, trebled, etc.?

How does the graph show the cost to vary as the weight is varied?

What is the ratio of the cost to the weight?

5. Write the equation for the weight w in pounds, and the cost c of flour at $3\frac{1}{2}$ cents a pound. Find the values of c as w takes the integral values from 1 to 10.

What is the ratio of the values of w to the corresponding values of c?

6. Graph the areas of rectangles with the altitude 5 and bases 1, 2, 3, ... 6. How does the area vary as the base varies?

Write the equation for the area A and the base b of rectangles with altitude 5.

Does a change in value of b change the value of A?

Does it change the value of the ratio $\frac{A}{b}$?

102. The number x is said to *vary directly* as the number y, if the ratio, $x:y$, remains constant (i. e., does not change).

1. Graph the areas of triangles with altitude 6 and bases 1, 2, 3, ... 7. Show that the area A of a triangle varies directly as the base b.

2. Write the equation for the area A of a triangle having the altitude h and the base b. Show from the equation that the area A varies directly as the base b, when the altitude is constant.

3. Write the equation for the area A of a rectangle having the altitude a and the base b. Show that the area A varies directly as the base b, when the altitude is constant.

4. Cut (from card-board) circles with diameters of various lengths. With a string or thin wire measure the circumferences of the circles. Find the ratio of the circumference of each circle to the diameter. How do the ratios compare?

5. The area of a rectangle varies directly as the base if the altitude remains constant; and when the area is 27, the base is 3. What is the constant ratio of the area to the base?

What is the equation connecting the area A and the base b? When the area is 54, what is the base?

6. The circumference of a circle varies directly as the diameter. The constant ratio of the circumference to the diameter is 3.14, approximately. Write the equation for the circumference c and the diameter d. When the circumference is 157, what is the diameter?

7. The distance d through which a body falls from rest varies directly as the square of the time t in which it falls; and a body is observed to fall 400 ft. in 5 seconds. What is the constant ratio of d to t^2?

Write the equation for d and t.

How far does a body fall in 1 second? In 2 seconds? In 3 seconds?

8. x varies directly as y, and when $x=20$, $y=4$. Find the value of x when $y=17$.

9. If z varies as x, and $z=48$ when $x=4$; find z when $x=11$.

10. The turning-tendency caused by a weight moved along a bar, or lever, varies directly as the lever-arm. The turning-tendency is 20 when the arm is 5. Find the turning-tendency when the arm is 7.

11. Rectangles with area 6 sq. ft. have bases of 1, 2, 3, ... 6. For each rectangle write the ratio of the base to the reciprocal of the altitude. How do the ratios compare in value?

103. The number b is said to *vary inversely* as the number a, if the ratio of b to the reciprocal of a, is constant.

1. Rectangles have an area equal to 10. The base varies inversely as the altitude. What is the constant ratio of the base to the reciprocal of the altitude? Write the equation for the base b and the altitude a.

2. y varies inversely as x. $y=2$ when $x=1$. Find the constant ratio of y to the reciprocal of x. Write the equation for x and y. Find y when $x=8$.

3. When gas in a cylinder is exposed to pressure, the volume is reduced as the pressure is increased. It is found in physics that the volume varies inversely as the pressure.

The volume of a gas is 4 cubic cm., when the pressure is 3 pounds. What is the volume under a pressure of 6 pounds?

4. The speed of a falling body varies directly as the time. Write the equation for the speed v and the time t.

A body, falling from rest, moves at the rate of 160 ft. a second 5 sec. after it began to fall. What will be the speed attained in 8 seconds?

5. The distance passed over by a body, moving at a constant rate, varies directly as the time.

Find the rate of a train which travels, at uniform rate, the distance of 225 mi. in 6 hours.

6. A stone fell from a building 560 ft. high. In how many seconds did it reach the ground? (See 7, § 102.)

7. The number of men doing a piece of work varies inversely as the time.

Twelve men can do a piece of work in 28 days. In how many days can 3 men do the same?

8. The time t of oscillation of a pendulum varies directly as the square root of the length l. A pendulum 39.2 in. long makes one oscillation in one second. Find the length of a pendulum which makes an oscillation in two seconds.

Summary

1. The **ratio** of two numbers is their *quotient.*

2. **Similar triangles** are triangles that have the *same shape.* Similar triangles are not necessarily of the same size.

3. All similar triangles may be regarded as the same triangle drawn to different scales. They may be regarded as the same triangle *magnified* or *minified* to a definite scale.

4. *Two triangles are similar when the corresponding angles are equal and when the ratios of the corresponding sides are equal.*

5. When surveying problems are solved by drawing triangles to scale, the triangle in the drawing is similar to the surveyed triangle.

6. A **proportion** is an *equality* of *ratios.*

7. Areas of rectangles (and triangles) have the same ratio
   (1) as the bases, if the altitudes are equal,
   (2) as the altitudes, if the bases are equal.

8. Areas of rectangles (and triangles) have the same ratio as the products of the bases and altitudes.

9. The **extremes** of a proportion are the first and last terms; the **means** are the second and third terms.

10. *In any proportion the product of the extremes is equal to the product of the means.* This is a convenient test of proportionality.

11. *If the product of two numbers is equal to the product of two other numbers, either pair may be made the means and the other pair the extremes in a proportion.*

12. A proportion is obtained from a given proportion by alternation, if the *means* are *interchanged,* or if the *extremes* are *interchanged.*

13. One of two numbers **varies directly** as the other if their *ratio* is *constant.*

14. One number **varies inversely** as another if the *ratio* of the first to the *reciprocal* of the second is *constant.*

CHAPTER VII

PROBLEMS ON PARALLEL LINES. GEOMETRIC CONSTRUCTIONS

Parallel Lines

104. The equation may be used to solve exercises on the angles made by lines intersecting parallel lines.

1. Point out the parallel edges of a rectangular box or table-top. Give other examples of parallel lines, as telegraph wires, latitude lines, plumb-lines, etc.

2. Draw two parallel straight lines, using the parallel edges of a ruler; draw a straight line intersecting them (Fig. 152). Measure and compare angles a and b, c and d, e and f, g and h. These angle-pairs are called pairs of **corresponding** angles.

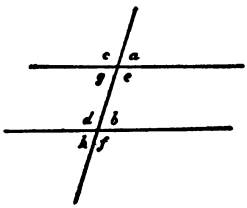


FIG. 152

3. Draw two figures like Fig. 152, with the crossing-lines in different directions, and compare the corresponding angles.

4. Draw two straight lines that are not parallel, and compare the corresponding angles that they make with a crossing-line (transversal).

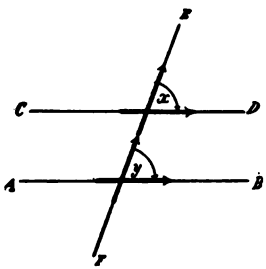


FIG. 153

105. Parallel straight lines are lines that have the same direction or opposite directions.

1. x and y (Fig. 153) are equal. Show by rotating a pencil, as indicated, that AB and CD have the same direction.

2. Draw two intersecting lines, A B and R S (Fig. 154). Show how to draw with the aid of a protractor a line parallel to A B through M.

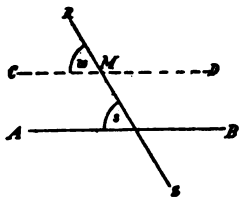


FIG. 154

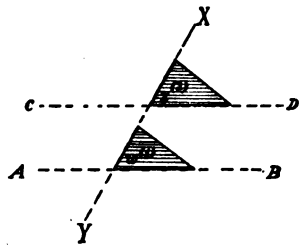


FIG. 155

3. Draw a triangle on paper. Cut out the triangle and place one side of it along a line X Y (Fig. 155), first in position (1) drawing line A B, and then in position (2) drawing C D. Show that A B and C D are parallel. What angles are equal?

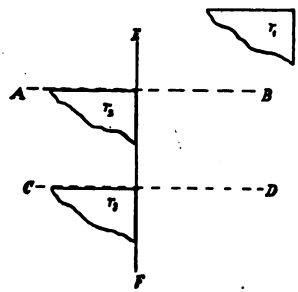


FIG. 156

4. Fold a piece of paper so as to form a right angle (r_1, Fig. 156). Using the right angle as in the figure, draw A B and C D and show that they are parallel.

106. In the preceding problems, the following properties of parallel lines were studied:

(*a*) If two straight lines are parallel, the corresponding angles made with a crossing-line (transversal) are equal.

(*b*) Two straight lines that make equal corresponding angles with a transversal have the same direction, and are parallel.

From (*b*) it follows that

(*c*) Two lines that are perpendicular to the same line are parallel.

1. In Fig. 157 read several pairs of supplementary adjacent angles (see p. 42); of vertical angles (see p. 47).

2. Draw two parallel straight lines and a transversal. Letter the angles as in Fig. 157. Give reasons for the following:

$$a=e \qquad (1)$$
$$a+d=180 \qquad (2)$$
therefore, $$e+d=180. \qquad (3)$$

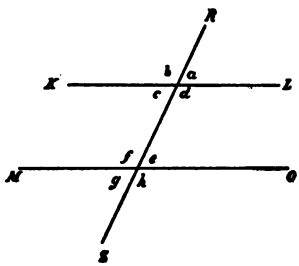


FIG. 157

Give reasons:

3. If $a=e$, then $c=e$ (Fig. 157).
4. If $c=e$, then $c=g$.
5. If $c=g$, then $a=g$.

6. Show that b is the supplement of a; that f is the supplement of e. Show that $b=f$, if $a=e$.

7. Give reasons for the following:

$$b=f \qquad (1)$$
$$b+c=180 \qquad (2)$$
therefore, $$f+c=180. \qquad (3)$$

Give reasons:

8. If $b=f$, then $d=f$.
9. If $d=f$, then $d=h$.
10. If $d=h$, then $b=h$.

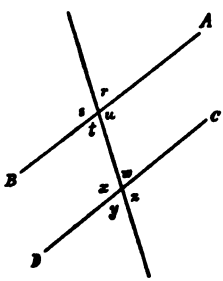


FIG. 158

11. Make a list of all equal angles of Fig. 157, and of the pairs of non-adjacent angles that were proved to be supplementary.

12. Make a list like 11 for Fig. 158, A B and C D being parallel.

13. One of the 16 angles made by the rails of two intersecting railroads is 35°. Find each of the other 15.

107. When two lines (Fig. 159) are cut by a transversal,

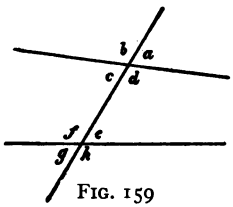


FIG. 159

the angles of the angle-pairs $\left\{\begin{matrix} a \text{ and } e \\ b \text{ and } f \\ d \text{ and } h \\ c \text{ and } g \end{matrix}\right\}$ are called **corresponding** angles,

angles c, d, e, f are called **interior** angles,

angles a, b, g, h are called **exterior** angles,

the angles of the angle-pairs $\left\{\begin{matrix} d \text{ and } e \\ c \text{ and } f \end{matrix}\right\}$ are called **interior** angles on the *same side* of the transversal,

the angles of the angle-pairs $\left\{\begin{matrix} d \text{ and } f \\ c \text{ and } e \end{matrix}\right\}$ on *opposite* sides of the transversal, are called **alternate interior** angles,

the angles of the angle-pairs $\left\{\begin{matrix} b \text{ and } h \\ a \text{ and } g \end{matrix}\right\}$ on *opposite* sides of the transversal, are called **alternate exterior** angles.

1. State the results of problem 11, p. 139, naming the angle-pairs as just defined.

2. Draw a pair of parallels crossing another pair. Letter the angles as in Fig. 160. Prove $a=t$, $t=m$; therefore $a=m$.

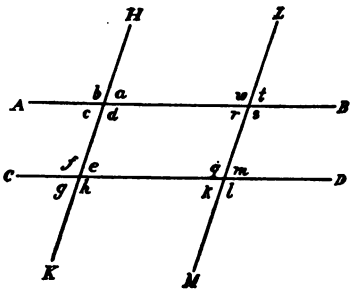


FIG. 160

3. Prove that $a+q=180°$.

$m+q=180°$. Why? $m=a$. Why? $a+q=180°$. Why?

4. Prove that r is the supplement of d, and that e is the supplement of d; therefore, $r=e$.

5. Prove the following: Fig. 160.

(1)	$d=q$	(5)	$g=t$
(2)	$w=f$	(6)	$c+q=180$
(3)	$w+e=180$	(7)	$a+l=180$
(4)	$b=l$	(8)	$s+e=180$.

6. From the top of a cliff (Fig. 161), 200 ft. high, the angle of depression of a buoy is 60°. Prove that angle B is 60°. Find, by a scale drawing, the distance of the buoy from the bottom of the cliff.

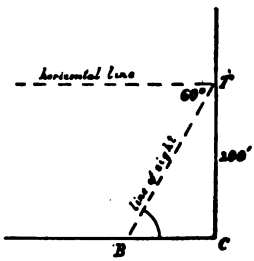


FIG. 161

108. A quadrilateral whose *opposite* sides are *parallel* is a **parallelogram.**

1. In the parallelogram, Fig. 162, prove that consecutive angles, as x and y, y and z, are supplementary.

2. Prove that the opposite angles, x and z, y and w, of the parallelogram, Fig. 162, are equal.

3. Prove that the sum of the interior angles of a parallelogram is four right angles.

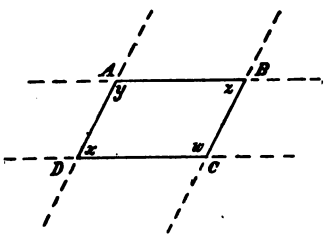


FIG. 162

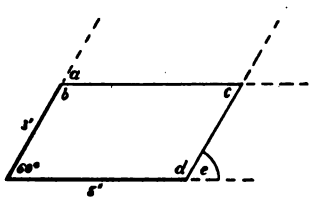


FIG. 163

4. With ruler and protractor draw a parallelogram having adjacent sides 3 in. and 5 in. (Fig. 163), and included angle 60°. How many degrees are there in a? In e? In b? In c? In d?

5. Draw parallelograms having the following parts. Find the number of degrees in each of the remaining interior angles:

	Adjacent Sides	Included Angle
(1)	3 in. and 5 in. and	120°
(2)	3 in. and 4 in. and	90°
(3)	3 in. and 3 in. and	40°
(4)	3 in. and 3 in. and	90°

109. The following theorems have been studied:

THEOREM I. *Two consecutive angles of a parallelogram are supplementary, and the opposite angles are equal.*

THEOREM II. *The sum of the interior angles of a parallelogram is four right angles.*

110. A quadrilateral with *one* pair of parallel sides is a **trapezoid.**

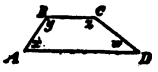


FIG. 164

1. In the trapezoid, Fig. 164, prove that x and y are supplementary.

2. Prove that the sum of the interior angles of a trapezoid equals four right angles.

3. Draw a trapezoid A B C D from the data of Fig. 165. Prolong B C and A D until they meet at O. Show that the angles of triangle O C D are equal to the corresponding angles of triangle O B A, and that the two triangles are similar.

4. In Fig. 165, show that O C:O B=O D:O A, and that C D:B A=O C:O B.

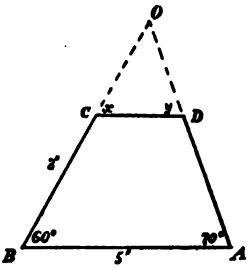


FIG. 165

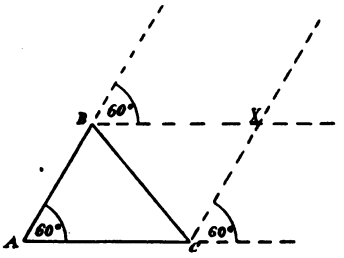


FIG. 166

5. Draw a triangle A B C (Fig. 166) having an angle of 60°. Draw lines B X and C X, making angles as indicated. Prove that A B X C is a parallelogram.

6. Draw a parallelogram, starting with a triangle having an angle of 135°; an angle of 90°.

7. Using Fig. 167, in which D E is parallel to A C, prove that the sum of the interior angles of a triangle is two right angles.

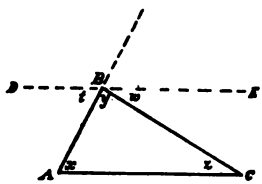


FIG. 167

8. What is the sum of the exterior angles (formed by prolonging a side) of a triangle taking one at each vertex?

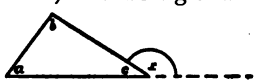


FIG. 168

9. From Fig. 168, prove that an exterior angle of a triangle (formed by prolonging a side) equals the sum of the interior angles not adjacent to it.

Algebraic Exercises on Geometric Figures

1. Two parallels and a transversal make angles that may be designated as shown in Fig. 169. Find the values of x, and of all the unknown angles.

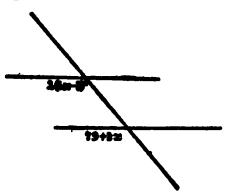


FIG. 169

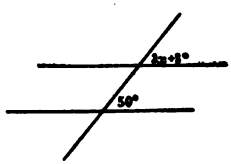


FIG. 170

2. Two parallels and a transversal form angles that may be designated as shown in Fig. 170. Find x and all the 8 angles.

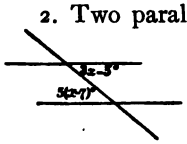


FIG. 171

3. With parallels, and angles as shown in Fig. 171, find x and all the 8 angles.

4. Two parallels are cut by a transversal making a pair of alternate interior angles of $5y+3°$ and $59°-3y$. Find y and all the 8 angles.

5. With two parallels and a transversal, a pair of corresponding angles are $x+2y$ degrees and $2(x-y)$ degrees, the angle adjacent to the latter being $120°$. Find x, y, and the unknown angles.

6. With two parallels and a transversal, a pair of alternate exterior angles are $5y-2x$, and $9x+y$. The angle adjacent to the latter is $86°$. Find x, y, and the unknown angles.

7. With two parallels, the interior angles on the same side of a transversal are $6x+y$ degrees and $14x-y$ degrees and their difference is $14°$. Find x, y, and all the 8 angles.

8. With two parallels, the interior angles on the same side of the transversal are $4x-y$ and $5(2y+x)$. $5(2y+x)$ and $125°$ are alternate interior angles. Find x, y, and the unknown angles.

9. With parallels, transversal, and angles as shown in Fig. 172, find x, y, and all the 8 angles.

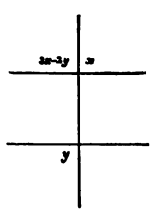


FIG. 172

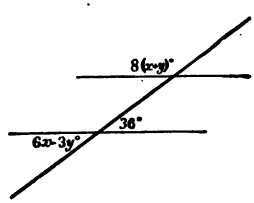


FIG. 173

10. With parallels, transversal, and angles as shown in Fig. 173, find x, y, and all the 8 angles.

11. With two parallels and a transversal, the alternate exterior angles are $7(x+1)$ degrees and $181-2x$ degrees. Find x and all the 8 angles.

Problems on Construction

111. The following geometrical figures are to be constructed by means of compasses and unmarked straight edge only.

1. *To construct a line perpendicular to a given line,* AB, *at a given point,* C, *on the line* AB.

CONSTRUCTION: With C (Fig. 174) as center and a convenient radius, draw arcs of a circle meeting A B in two points as D and E.

With D and E as centers and a convenient radius, draw arcs of two circles meeting as at F and G.

Connect by a straight line one of the points as F with C.

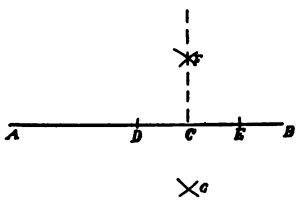


FIG. 174

The line F C is the required perpendicular to A B at point C.

How long must the radius D F be?

2. Construct a right triangle having given the two sides, 3 in. and 4 in., which meet at the vertex of the right angle.

A **rectangle** is a *parallelogram* one of whose angles is a right angle.

3. To construct a rectangle, having given the length l and the width w.

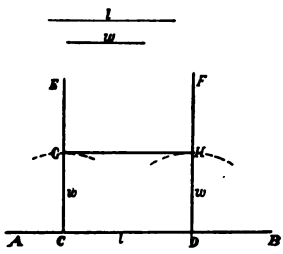


FIG. 175

CONSTRUCTION: On any convenient line, A B (Fig. 175), lay off C D equal in length to l.

At C and D construct the perpendiculars C E and D F.

On C E and D F lay off the lengths C G and D H each equal to w.

Draw G H.

The quadrilateral C G H D is the required rectangle.

4. Construct the complement of a given acute angle A B C.

5. Construct a square having given one side s.

6. Construct two lines perpendicular to a given line and prove the two lines parallel.

7. Draw a right triangle. Construct a rectangle having for two of its sides two sides of the triangle.

8. *From a point,* C, *outside of a line,* A B, *to construct a line perpendicular to* A B.

Construct C F (Fig. 176) perpendicular to A B, following the directions given in the construction for problem 1.

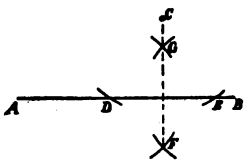


FIG. 176

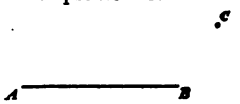


FIG. 177

9. Construct a perpendicular from C to A B when C and A B are as in Fig. 177.

10. Draw a triangle of which all angles are acute. From the vertex of each angle construct a line perpendicular to the opposite side.

It is proved in geometry that the three altitudes of a triangle meet in a point. This gives a test of the accuracy of the constructions in problems 10, 11, and 12. Crease the three altitudes of a paper triangle and find whether they all pass through the same point.

11. Draw a triangle having an obtuse angle and construct perpendiculars as in problem 10.

12. Draw a right triangle and construct perpendiculars as in problem 10.

13. Draw a parallelogram, as A B C D (Fig. 178). Construct the altitudes from B and C to the base A D; from D to the base A B.

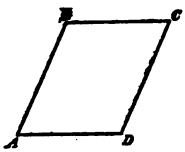


FIG. 178

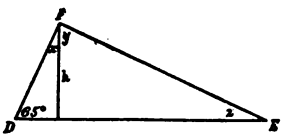


FIG. 179

14. Angle F (Fig. 179) is a right angle and h is the altitude from F to D E. Find the values of x, y, and z.

15. *To bisect a given line,* A B.

CONSTRUCTION: Using the end points A and B as the points D and E were used in problem 1, draw a line, as F G, perpendicular to A B.

The point of intersection H is the required midpoint of A B.

The line F G is called the **perpendicular bisector** of A B.

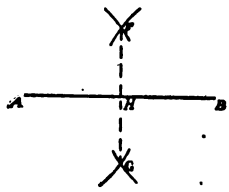


FIG. 180

16. Draw a triangle. Bisect the sides and join the midpoint of each side to the vertex of the opposite angle.

A straight line drawn from the midpoint of a side of a triangle to the vertex of the opposite angle is a **median.**

Show, by creasing the 3 medians of a paper triangle, how the accuracy of the construction of the three medians of a triangle may be tested.

17. Draw a right triangle. Construct the medians and show, by drawing a circle on the hypotenuse as a diameter, that the median to the hypotenuse equals one-half of the hypotenuse.

18. Draw a right triangle, a triangle having an obtuse angle, and a triangle of which all angles are acute. In each of the triangles construct the perpendicular bisectors of the three sides. What seems to be the test for the accuracy of the construction? Crease the perpendicular bisectors of the three sides of a paper triangle and see whether the test is met.

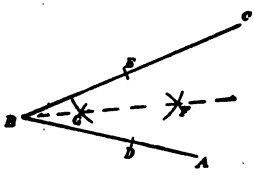


FIG. 181

19. *To bisect a given angle* A B C.

CONSTRUCTION: With B as center and any radius, draw arcs intersecting B A and B C in two points as D and E.

With D and E as centers and a convenient radius, draw arcs meeting as at F and G.

Connect by a straight line one of these points, as F, with B.

The line B F is the required bisector of angle A B C.

20. Draw a triangle. Construct the bisectors of the three angles of the triangle. What seems to be a test for the accuracy of the construction? Find a test for the accuracy of the construction by creasing the bisectors of the angles of a paper triangle.

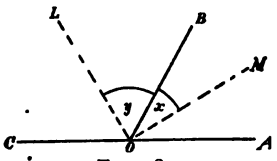


FIG. 182

21. Construct the bisectors (L O and M O) of two supplementary adjacent angles (Fig. 182). Prove that the bisectors are perpendicular to each other.

22. Construct the bisectors of a pair of corresponding angles made by a transversal cutting two parallels, Fig. 183. Prove that the bisectors are parallel.

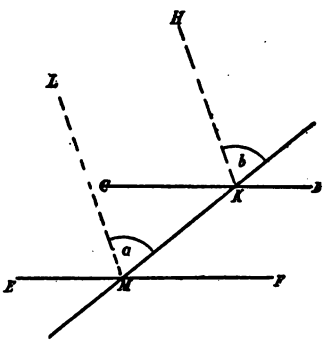


FIG. 183

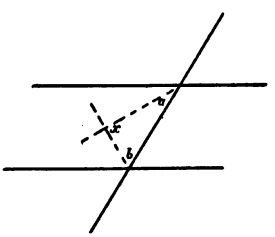


FIG. 184

23. Construct the bisectors of a pair of interior angles on the same side of a transversal cutting two parallels (Fig. 184). Prove that the bisectors are perpendicular to each other.

24. *At a given point,* A, *on a line,* B C, *to construct an angle equal to a given angle,* D E F.

CONSTRUCTION: With E as center (Fig. 185) and any radius, draw an arc meeting E F and E D in the points, G and H.

With A as center and the same radius, draw the arc M N.

With M as center and radius equal to the distance from H to G, draw an arc meeting M N at O.

Draw A O.

Then O A M is the required angle.

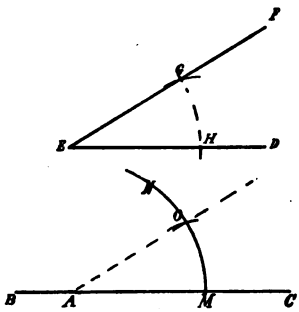


FIG. 185

25. Draw a line parallel to a given line passing through a point outside of the given line.

Construction is as in problem 2, p. 138. Instead of using the protractor to construct the angle, ruler and compasses are to be used.

26. Draw a triangle. Construct an angle equal to the sum of the angles of the triangle. How can you test the accuracy of the construction?

27. Draw a triangle. Construct a parallelogram having for two of its sides two sides of the triangle.

28. To construct triangles having the same base and equal areas.

CONSTRUCTION: Construct a line C D (Fig. 186) parallel to A B.

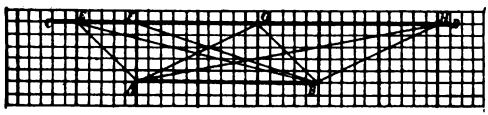


FIG. 186

Connect, by straight lines, the points A and B with points on C D as E, F, G, etc.

Show that the triangles thus formed have equal altitudes and equal areas.

29. Draw a triangle. Construct another triangle having two sides and the included angle equal to two sides and the included angle of the first. Compare the triangles as to size by cutting out one of them and fitting it on the other.

30. Draw a triangle. Construct another triangle having one side and the two angles adjacent to that side equal to a side and its adjacent angles of the first. Compare the triangles as to size.

Algebraic Exercises on Geometric Figures

1. The angles made by two pairs of parallels intersecting as in Fig. 187 are designated as shown. Find x, y, and all 4 angles about the crossing-point, K.

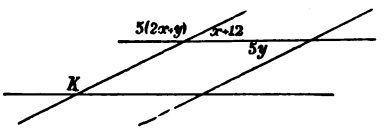


FIG. 187

2. With two pairs of intersecting parallels and angles as shown in Fig. 188, find x, z, and all 4 angles around any crossing-point.

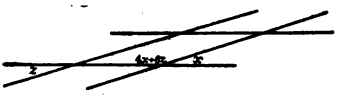


FIG. 188

3. With the sides and angles as shown in Fig. 189, find y, z, a, and b, and all the 4 angles and 4 sides of the parallelogram. Assume that the opposite sides of a parallelogram are equal.

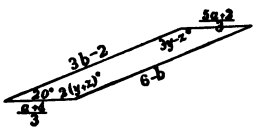


FIG. 189

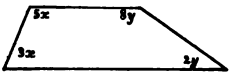


FIG. 190

4. In the trapezoid of Fig. 190, find x, y, and all 4

5. In the trapezoid, Fig. 191, find x, y, and all 4 interior angles.

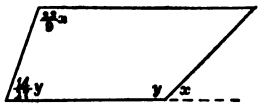


FIG. 191

6. The non-parallel sides of a trapezoid are extended until they meet, Fig. 192. The lengths of lines being designated as shown, find x and y.

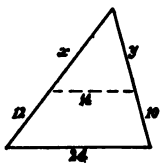


FIG. 192

7. The non-parallel sides of a trapezoid being prolonged to intersect and the lengths of lines being designated as shown in Fig. 193, find x and y and the lengths of all lines.

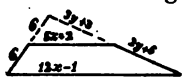


FIG. 193

8. The non-parallel sides of a trapezoid are extended to meet, Fig. 194. Find the lengths of lines, b, c, and the area of the trapezoid in terms of a.

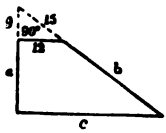


FIG. 194

9. Find all angles of Fig. 194, lengths of lines being as shown and the angle at the top, 53° 8′.

SUMMARY

1. **Parallel** straight lines are lines that have the same direction or opposite directions.

2. *If two lines are cut by a transversal, making a pair of corresponding angles equal, the lines are parallel.*

3. *Two lines that are perpendicular to the same straight line are parallel.*

4. *If two parallel lines are cut by a transversal,*
   (1) *the corresponding angles are equal;*
   (2) *the alternate-interior angles are equal;*
   (3) *the sum of the interior angles on the same side of the transversal is two right angles.*

5. A **parallelogram** is a *quadrilateral* whose opposite sides are *parallel.*

6. *Two consecutive angles of a parallelogram are supplementary, and the opposite angles are equal.*

7. The sum of the interior angles of a parallelogram is four right angles.

8. *An exterior angle of a triangle (formed by prolonging a side) is equal to the sum of the two interior angles not adjacent to it.*

9. A **trapezoid** is a *quadrilateral* with *one* pair of parallel sides.

10. The sum of the interior angles of a trapezoid is four right angles.

11. To construct figures by the aid of compasses and unmarked straight edge only, the following fundamental constructions are used:

(1) *To construct a perpendicular to a line at a point on the line;*

(2) *To construct a perpendicular to a line from a point without;*

(3) *To bisect a line;*

(4) *To bisect an angle;*

(5) *To construct an angle equal to a given angle.*

12. A **median** of a triangle is a straight line drawn from the *mid-point* of a side to the vertex of the opposite angle.

CHAPTER VIII

THE FUNDAMENTAL OPERATIONS APPLIED TO INTEGRAL ALGEBRAIC EXPRESSIONS

Addition of Monomials

112. The tickets for a football game are sold at 25¢ by John, Henry, Kenneth, William, and James. They report sales as follows: John sold 56 tickets, Henry 75, Kenneth 27, William 83, James, 69. At the gate 123 tickets are sold. Find the total receipts.

SOLUTION I:

John,	$56\times25¢=$	\$14.00
Henry,	$75\times25¢=$	18.75
Kenneth,	$27\times25¢=$	6.75
William,	$83\times25¢=$	20.75
James,	$69\times25¢=$	17.25
Gate,	$123\times25¢=$	30.75
	Total receipts	\$108.25

SOLUTION II:

John,	56 tickets,	$56\times25¢$
Henry,	75 tickets,	$75\times25¢$
Kenneth,	27 tickets,	$27\times25¢$
William,	83 tickets,	$83\times25¢$
James,	69 tickets,	$69\times25¢$
Gate,	123 tickets,	$123\times25¢$
	433 tickets,	$433\times25¢=\$108.25.$

In SOLUTION II the addition is simplified, because the different terms to be added have a common factor and they were therefore added by adding the *coefficients* of this factor in the different terms.

1. The tickets being sold at $x¢$, John sells 60 tickets, Henry 78, Kenneth 45, William 36, and James 84. At the gate 137 tickets are sold. Find the total receipts.

Total receipts: $\underbrace{60x}_{\text{John}}+\underbrace{78x}_{\text{Henry}}+\underbrace{45x}_{\text{Kenneth}}+\underbrace{36x}_{\text{William}}+\underbrace{84x}_{\text{James}}+\underbrace{137x}_{\text{Gate}}=440x.$

Again the different terms to be added had a common factor and the polynomial resulting from the addition could be simplified by adding the coefficients of the common factor in the different terms.

Is SOLUTION I of p. 153 possible in this case? Give a reason for your answer.

2. I wish to change the following German currency to American money: 10 five-mark pieces, 17 one-mark pieces, 28 twenty-five-mark bills, and 123 fifty-mark bills. The value of a mark is 23.8¢. How much American money should I receive in exchange?

3. The length of the school hall is s feet. I go through the hall 6 times on Monday, 8 times on Tuesday, 4 times on Wednesday, 6 times on Thursday, and 10 times on Friday. How many feet do I travel along the hall during the week?

4. The running track in the playground is y yards. While in training, I take it 6 times on Monday, 8 times on Tuesday, 10 times on Wednesday, 12 times on Thursday, and 14 times on Friday. How many yards do I run during the week?

113. Terms which have a *common factor* are called *similar with respect to that factor.*

1. Point out with respect to what factor the following terms are similar. State in each case the coefficient of the common factor:

(1) $4x, -7x, 20x, -35x$
(2) $ax, -25x, -bx, 46x$
(3) $ax, -bx, -cx, dx$
(4) $2xa, 3xa, -7xa, -5xa$
(5) $-3pq^2, 6tq^2, -8rq^2, 12sq^2$
(6) $4axz, -7cxz, -5dxz, 9exz$
(7) $abm, -pmq, -xmy, -mdc$

(8) $3a^2b$, $-1\frac{1}{2}ab^2$, $+4\frac{1}{2}a^2b^2$, $-1.5a^3b$
(9) ax^2, $3\frac{1}{2}xy$, $-7.5xt$, $-17bx^2$
(10) $-4at^3$, $-abt^2$, $+3axt^2$, $+7.2ayt^2$.

2. State a rule for simplifying polynomials consisting of similar terms.

Exercise XV

Express the sums of the following numbers in the form of a polynomial and simplify:

1. $+15a$, $-7a$, $+18a$
2. $-18x^2$, $-12x^2$, $+15x^2$, $-3x^2$
3. $+2\frac{1}{2}ab$, $+3\frac{1}{3}ab$, $-4\frac{1}{2}ab$, $-5\frac{2}{3}ab$
4. $+27abc$, $-35abc$, $+10abc$, $-2abc$
5. $3(a+b)$, $-4(a+b)$, $12(a+b)$
6. $-8(x^2+y^2)$, $-24(x^2+y^2)$, $17(x^2+y^2)$
7. $-3\frac{2}{5}(pr-q^2)$, $+5\frac{2}{3}(pr-q^2)$, $-4\frac{7}{10}(pr-q^2)$
8. $18(mp-3s)^2$, $-15(mp-3s)^2$, $-37(mp-3s)^2$, $14(mp-3s)^2$
9. $a(x+y+z)$, $-b(x+y+z)$, $-c(x+y+z)$, $d(x+y+z)$
10. $15ax^2$, $-7bx^2$, $8dx^2$, $-5cx^2$.

114. At a money changer's the following amounts of money are offered for exchange: 527 marks, 349 francs, 57 pounds. The exchange value of a mark, a franc, and a pound are 23.8¢, 19.3¢, and \$4.87 respectively. What is the exchange value of the foreign currency.

527 marks	at 23.8¢ = 527 × 23.8¢ =	\$125.426
349 francs	at 19.3¢ = 349 × 19.3¢ =	67.357
57 pounds	at \$4.87 = 57 × \$4.87 =	277.59
		\$470.373

Could the method of Solution II on p. 153 have been applied here?

1. A beam is loaded as indicated below. Find the total turning-tendency in each case:

	f_1	d_1	f_2	d_2	f_3	d_3	f_4	d_4
I	$+3$	-17	-5	-14	-8	-5	$+12$	$+19$
II	$+5$	$-a$	$+12$	$-b$	$+14$	$+b$	-6	$+a$
III	-8	$-x$	$+10$	$-y$	-15	$-z$	$+9$	$+w$

2. The main stairway in a school consists of four flights of stairs, having a, b, c, and d steps, respectively. If a pupil goes up and down 8 times in one day, how many steps does he take on that stairway daily?

115. Terms which have *no common factor* are called **dissimilar terms.**

Can a polynomial, whose terms are not all similar, be simplified into one term?

Exercise XVI

Add the following numbers and simplify the resulting polynomials as far as possible.

1. $5s^2t$, $-12x^2y$, $+7x^2y$, $-3s^2t$
2. $-27\frac{2}{3}ab$, $+18\frac{1}{3}cd$, $15\frac{1}{5}ab$, $14\frac{2}{5}cd$
3. $5ax$, $-3x$, x
4. $9a^2b^2$, $-3c^3y^3$, $4a^2b^2$, $-4c^3y^3$, $-3a^2b^2$
5. $3mp^2$, $-8mp^2$, $+5a^2x$, $-3a^2x$, $-4mp^2$, $2a^2x$
6. $-4st$, $-t$, $3t$, $+5st$
7. ar^2p, $-r^2p$, $-br^2p$, pr^2, $-cpr^2$
8. $27(a+b)^2$, $+4c$, $+15a$, $-12c$, $-15(a+b)^2$, $-8a$
9. $a(a-b)$, $b(a-b)$, $-c(a-b)$
10. $a^2(a+b)$, $-2ab(a+b)$, $b^2(a+b)$
11. $-6\frac{1}{3}(a+b)$, $\frac{1}{2}(a^2+b^2)$, $-\frac{1}{4}(a^2+b^2)$, $+8\frac{2}{3}(a+b)$, $+4\frac{2}{3}(a+b)$, $6\frac{1}{2}(a^2+b^2)$
12. $r^3(r+p)$, $-3r^2(r+p)$, $3r(r+p)$, $r+p$.

Addition of Polynomials

116. Consider problem 2, p. 156. If a pupil goes up and down the stairs 5 times on Monday, 6 times on Tuesday, 4 times on Wednesday, 5 times on Thursday, and 4 times on Friday, how many steps does he take on the stairway every week?

On Monday:	$10a+10b+10c+10d$
On Tuesday:	$12a+12b+12c+12d$
On Wednesday:	$8a+8b+8c+8d$
On Thursday:	$10a+10b+10c+10d$
On Friday:	$8a+8b+8c+8d$
	$48a+48b+48c+48d$

We add these 5 polynomials by adding the similar terms in them.

1. Add:

$$\begin{array}{r} +27x^2-15xy+18y^2 \\ \underline{-12x^2+30xy-3y^2} \end{array}$$

Adding the similar terms in the polynomials:

$$+15x^2+15xy+15y^2.$$

Frequently the work is arranged along one line as follows:

$$(+27x^2-15xy+18y^2)+(-12x^2+30xy-3y^2).$$

Adding similar terms: $15x^2+15xy+15y^2$.

Exercise XVII

Add the following polynomials and simplify:

1. $5a+3b+9c,\ 2a-10b-2c$
2. $6a+15c-17b-8d,\ -7a+21d+15b-12c$
3. $7a^2+23b^2-15c^2,\ -21b^2-6a^2+12c^2$

*4. $29xy-7y^2+24x^2,\ -14xy-36x^2+36y^2$

5. $-3a-7b+14c,\ -11a+20b-34c$

*6. $14k-11l+12m,\ -3k+12l-6m,\ -12k+l-2m$

7. $-18a^2b^2+12a^4-8a^3b,\ 4\frac{1}{2}a^3b-a^2b^2+3\frac{3}{8}a^4$
8. $\frac{5}{8}x^2-\frac{1}{3}xy-\frac{1}{4}y^2,\ -x^2-\frac{2}{3}xy+2y^2,\ \frac{2}{5}x^2-xy-5y^2$
9. $5(a+b)-7(a^2+b^2)+8(a^3+b^3),\ -4(a^3+b^3)+5(a^2+b^2)-5(a+b)$

*10. $\frac{3}{2}(a+b+c)-\frac{3}{4}(a-b+c)+\frac{4}{5}(a+b-c)+1\frac{1}{5}(-a+b+c),$
$-\frac{2}{3}(a+b+c)+\frac{4}{3}(a-b+c)-\frac{5}{4}(a+b-c)-\frac{5}{6}(-a+b+c)$

*The exercises marked with a * should be arranged and solved in both ways.

11. $p^3+3p^2+4p-6,\ -p^2-2p+1,\ p^2-1,\ 3p^3+2p+2$

*12. $x^2+2xy+y^2,\ x^2-2xy+y^2,\ x^2-4xy,\ 4xy+y^2$

13. $-23a^2b+41a^2c+56c^2b-15b^2c, -6a^2b+26a^2c+59c^2b$
$-26b^2c,\ 25a^2c+19b^2c-18c^2b$

14. $5a^3(a+b)-6a^2b(a^2+b^2)+3ab^2(a^3+b^3),$
$a^2b(a^2+b^2)+4a^3(a+b)-7ab^2(a^3+b^3)$

*15. $6(lr+t)+7(l-u)+tv,\ 5tv-8(lr+t)-5(l-u),$
$3(lr+t)-(l-u)-4tv.$

Subtraction of Monomials

117. It has been shown (see pp. 75, 76) how subtraction of algebraic numbers may always be changed into addition.

EXAMPLE: $\begin{array}{r} +5x \\ -7x \\ \hline \end{array}$ **S** may be replaced by $\begin{array}{r} +5x \\ +7x \\ \hline \end{array}$ **A**

Subtract the lower monomials from the upper:

1. $\begin{array}{r} +15x^2 \\ -17x^2 \\ \hline \end{array}$ 2. $\begin{array}{r} -7\frac{1}{2}a \\ +3\frac{1}{2}a \\ \hline \end{array}$ 3. $\begin{array}{r} -15\ ab \\ -18\frac{2}{3}ab \\ \hline \end{array}$ 4. $\begin{array}{r} -\ 8\frac{2}{3}(p+q) \\ +14\frac{1}{3}(p+q) \\ \hline \end{array}$

5. $\begin{array}{r} +5m^2px \\ -4m^2px \\ \hline \end{array}$ 7. $\begin{array}{r} -7\frac{2}{5}(x-y) \\ -5\frac{1}{3}(x-y) \\ \hline \end{array}$

6. $\begin{array}{r} +14x(1+5a^2y) \\ +\ 8x(1+5a^2y) \\ \hline \end{array}$ 8. $\begin{array}{r} +11m^2(a-2b^3) \\ +29m^2(a-2b^3) \\ \hline \end{array}$

118. Instead of writing the subtrahend under the minuend, it is often written on the same line with the minuend, connected with it by a minus sign, $-$, to indicate subtraction, thus:

$(+5x)-(-7x)$ is equivalent to $(+5x)+(+7x)$ or $+5x+7x$ or $+12x$. Why?

Omitting the second step the work may be written as follows:

$$(+5x)-(-7x)=+5x+7x=+12x.$$

* The exercises marked with a * should be arranged and solved in both ways.

Exercise XVIII

Simplify:

1. $(+5ab)-(+12ab)$
2. $(-75a^2rl^2)-(-54a^2rl^2)$
3. $(-18p^3f^3)-(+63f^3p^3)$
4. $(+25s^2gh)-(-75s^2gh)$
5. $(-3\frac{1}{2}hk^2v)-(-2.4hk^2v)$
6. $(+8.7p^3q^2s^4)-(-4\frac{1}{5}p^3q^2s^4)$
7. $\{-5(a+b)\}-\{+7(a+b)\}$
8. $\{+4(t^2-f^3)\}-\{-2(t^2-f^3)\}$
9. $\{-6(m^2+g)\}-\{-a(m^2+g)\}$
10. $\{-x(v^2-s^2)\}-\{+7(v^2-s^2)\}$
11. $\{a(x+y)\}-\{b(x+y)\}$
12. $\{-2\frac{1}{2}a(x^2-ay)\}-\{+4e(x^2-ay)\}$
13. $\{+18\frac{2}{5}(a+b+c)\}-\{+25\frac{1}{3}(a+b+c)\}$
14. $\{-3.4t^2(v^2-h^2)\}-\{-4.5s^2(v^2-h^2)\}$.

Subtraction of Polynomials

119. When the subtrahend consists of more than one term, the subtraction may be performed by subtracting each term of the subtrahend from the minuend.

For example, when we wish to subtract 7 dollars, 4 quarters, and 10 dimes from 15 dollars, 8 quarters, and 30 dimes, we subtract 7 dollars from 15 dollars, leaving 8 dollars; 4 quarters from 8 quarters, leaving 4 quarters; and 10 dimes from 30 dimes, leaving 20 dimes.

The subtraction of algebraic polynomials is then not different from the subtraction of monomials and may again be reduced to addition.

Example:

$$\begin{array}{r} +9x^2-14xy-12y^2 \\ -7x^2+\ 5xy-15y^2 \\ \hline \text{S} \end{array} \quad \text{may be replaced by} \quad \begin{array}{r} +9x^2-14xy-12y^2 \\ +7x^2-\ 5xy+15y^2 \\ \hline \text{A} \end{array}$$

1. Subtract the lower from the upper polynomial:

$$(1)\quad \begin{array}{r} a^2+2ab+b^2 \\ a^2-2ab+b^2 \\ \hline \end{array} \qquad (2)\quad \begin{array}{r} a^2b^2+a^4-b^4-3a^3b+4ab^3 \\ -a^2b^2-a^4-b^4-3a^3b-4ab^3 \\ \hline \end{array}$$

$$(3)\quad \begin{array}{r} x^3+3x^2y+3xy^2+\ \ y^3 \\ -3x^2y+3xy^2-3y^3 \\ \hline \end{array}.$$

2. Without rewriting, subtract the upper polynomial from the lower in the preceding problem.

3. From $c^3-2a^2c-d^3-r^2$ subtract $-a^2c-3d^3-r^2-c^3$.

4. From $-17x^4+16y^4+4x^3y-29xy^3-35x^2y^2$ subtract $+15y^4-41x^2y^2+8x^3y-18x^4-25xy^3$.

5. Again the subtraction of polynomials may be arranged as described in §**118**, p. 158.

EXAMPLE: Instead of writing:

$$\begin{array}{r} -\ 4a^2b+7ab^2-12a^3-25b^3 \\ -18a^2b+9ab^2+\ 5a^3-\ 4b^3 \\ \hline \end{array}$$ **S**, we write:

$(-4a^2b+7ab^2-12a^3-25b^3)-(-18a^2b+9ab^2+5a^3-4b^3)$, which is equivalent to:

$(-4a^2b+7ab^2-12a^3-25b^3)+(+18a^2b-9ab^2-5a^3+4b^3)$
or: $-4a^2b+7ab^2-12a^3-25b^3+18a^2b-9ab^2-5a^3+4b^3$
or: $-17a^3+14a^2b-2ab^2-21b^3$. Why?

Leaving out the second step, the work may be written:

$$\begin{aligned}(-4a^2b+7ab^2-12a^3-25b^3)-(-18a^2b+9ab^2+5a^3-4b^3)&= \\ -4a^2b+7ab^2-12a^3-25b^3+18a^2b-9ab^2-5a^3+4b^3&= \\ -17a^3+14a^2b-2ab^2-21b^3.&\end{aligned}$$

EXERCISE XIX

Simplify:

1. $(+17x^2-14xy-15y^2)-(-16x^2+12xy-9y^2)$
2. $(-5a^2x+10bxy+24b^2y-18axy)-(-6b^2y+12axy-4a^2x-2bxy)$
3. $(-9m^2pq-5m^3p-14m^2q^2)-(-6m^3p-10m^2pq-13m^2q^2)$

4. $(5\frac{2}{3}abc-7\frac{1}{2}a^2b-8\frac{2}{5}b^2c-6\frac{1}{3}c^2a)-(4\frac{1}{3}a^2b-5\frac{2}{3}c^2a$
$+3\frac{1}{5}abc+7\frac{1}{3}b^2c)$

5. $(45x^2y^2-27x^4+81y^4)-(73x^4+45y^4+65x^2y^2)$

6. $(+4x^3-3x^2y+12y^3-7xy^2)-(2y^3-3xy^2+4x^2y+7x^3)$

7. $(\frac{1}{2}l^3-3\frac{1}{2}lm^2+4\frac{1}{3}m^3-3l^2m)-(\frac{3}{4}lm^2+5\frac{1}{2}l^2m-2l^3-3m^3)$

8. $(3.4v^3s^4-5.7v^4s^3+9.8v^6s^2)-(-1.7v^4s^3-3.2v^6s^2$
$-4v^7s)$

9. $(27x^3-6x^2y+8y^3)-(-15x^3+8xy^2-4y^3)$

10. $(4t^4-7\frac{1}{2}f^3)-(3\frac{1}{2}t^3f-7.6t^4+5\frac{2}{3}t^2f^4+6f^3)$

11. $(-56g^4+27g^3h-14gh^3)-(-13g^3h+18g^2h^2-6gh^3$
$+25h^4)$

12. $(-2\frac{1}{2}k^3ml+7.5k^2m^2l-3.24km^3l)-(3\frac{1}{4}km^3l-3.6km^2l^2$
$-5.4k^3ml+6kml^3)$

13. $(3ax^3-4syz^3)-(-2bx^3+4x^3-5z^3+3tyz^3)$

14. $(-5mv^2-3mvu+4mu^2)-(+3v^2-6muv-4nuv$
$+9nu^2)$.

120. Compare the signs of the terms of the subtrahend in the separate exercises of EXERCISE XIX, before and after the parentheses are removed.

1. State a rule as to the effect of a minus sign in front of a polynomial in parentheses.

2. State a similar rule as to the effect of a plus sign.

EXERCISE XX

Perform the following operations and simplify results:

1. $4-\{5-(st^2-4)\}$

2. $8\frac{1}{2}-\{4k^3-\overline{3k^3-5\frac{1}{2}}\}$

3. $16e^2-\{-42e^2-\overline{3e^2+2}\}-(50e^2+3)$

4. $2f-[6f-\overline{3g-4f}-(2g-4f)]$

5. $4t^2-\{t^2-3t^3+\overline{3t^2-t^3}\}$

6. $9x-\{5y-(6y+7z)-(7y-4z)\}$

7. $-3a^4+[4a^3-(3a^3-5a^2)-(4a^2+3a)]$

8. $7.5p^3-[3.4p^3-\overline{4.2p^3+1.6p^2}-\overline{3.4p^2-4.5p}]$

9. $3k^3-\{2p^2+k^3-\overline{p^2+r}\}-[p^2-r-(k^3+r)]$

10. $105s^2-\overline{14st+3t^2}+\{4st-2t^2-(5s^2-2st)\}$

11. $-[3rl-(3r^2-3l^2)]-\{4r^2-\overline{5rl-2l^2}\}+(2r^2+2l^2)$

12. $-\{5\frac{1}{2}k^2-(3.4kh-2h^2)\}-[5.4kh-\overline{3k^2-2.4h^2}]$
$-(7k^2+2\frac{1}{2}kh-4h^2)$.

Multiplication of Monomials

121. The area of a rectangle and the surface and volume of a rectangular block can be easily found when the dimensions are known.

1. Find the areas of rectangles, whose dimensions are:

(1) 5 in. and 7 in.
(2) $13\frac{1}{2}$ cm. and $16\frac{2}{3}$ cm.
(3) 7 ft. and a ft.
(4) x m. and $9\frac{1}{2}$ m.
(5) p rd. and q rd.
(6) m^2 yd. and m yd.
(7) a^2b mi. and b^2c mi.
(8) $2y^3$ km. and $3y^2$ km.
(9) $5\frac{1}{3}x^2y$ ft. and $3\frac{3}{4}xy^2$ ft.
(10) $4.5abc$ in. and $3.2abc$ in.

Illustrate the problems by drawings on square-ruled paper whenever possible, and check results by counting the number of squares contained in each rectangle.

In the above and in all the following problems with literal numbers test the values obtained by assigning numerical values to the letters.

For example, the area of a rectangle of dimensions $3ab$ and $2bc$ is $6ab^2c$. Assigning to a, b, and c, the values 2, 3, and 4 respectively, the dimensions and area are 18, 24, and 432 respectively, which is correct, since $18\times24=432$.

2. Find the area of a square, whose edge is: 15 ft.; $8\frac{1}{5}$ m.; a^3 cm.; $7.5bc$ in.; x^6 in.; a^3b^2 mi.; $3\frac{1}{2}p^4q^5$ m.; $7x^3y^7z$ ft.; $4.32a^2b^5c^4$ dm.; $9mn^2p^3q^4$ cm.

3. Find the volume and area of a cube, whose edge is: $7\frac{1}{2}$ cm.; $2x^2$ ft.; $5\frac{1}{3}ab$ in.; $2.4y^3$ m.; $4xyz$ cm.; $5.6x^2y^3z^2$ mi.; $3\frac{2}{7}x^4$ rd.; $12a^2b^2$ yd.; $4\frac{1}{2}a^5$ ft.; $15p^3q^2r$ m.

4. Find the area and volume of a rectangular block, whose dimensions are:

(1) 4, 9, and 15 in.
(2) $3\frac{1}{3}$, $5\frac{1}{7}$, and x ft.
(3) x, 8, y m.
(4) a^2, a^3, a^4 rd.
(5) x^3, $2x^2$, $5x^5$ yd.
(6) $3.5p^3$, $4.5p^4$, $16p^5$ cm.
(7) x^2yz, xy^2z, xyz^2 ft.
(8) $2amp$, $3bpm$, $4cmp$ in.
(9) $4\frac{2}{3}x^8$, $2\frac{1}{7}x^{10}$, $\frac{1}{10}x^{12}$ m.
(10) $5x^3y^2$, $4\frac{1}{5}x^2y^3$, $\frac{3}{7}x^4y^4$ cm.

5. We have seen (p. 21), that in x^5, 5 is called the exponent, x is called the base, and x^5 is called the 5th power of x. Give other examples of different powers of the same base.

6. In what way are powers of the same base multiplied? Can powers of different bases be multiplied in the same way?

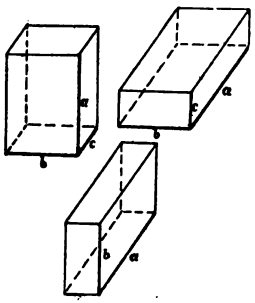


FIG. 195

7. What is the volume of a rectangular block of dimensions a, b, and c ft.? b, c, and a ft.? b, a, and c ft.? c, b, and a ft.? c, a, and b ft.? a, c, and b ft.? What is the effect of the *order* in which the factors are multiplied upon the value of the product?

8. What is the total volume of 7 rectangular blocks of dimensions x, y, and z? What is the volume of one rectangular block of dimensions $7x$, y, and z? x, $7y$, and z? x, y, and $7z$? $7x$, $7y$, and z? $7x$, y, $7z$? x, $7y$, $7z$? $7x$, $7y$, $7z$?

9. Which of the following expressions are equal?

(1) $7(x \cdot y \cdot z)$	(5) $5(4 \cdot 3 \cdot 6)$	(9) $8(7 \cdot 6 \cdot 5)$
(2) $7x \cdot 7y \cdot 7z$	(6) $5 \cdot 3 \cdot 5 \cdot 6 \cdot 4$	(10) $8 \cdot 7 \cdot 8 \cdot 6 \cdot 8 \cdot 5$
(3) $7x \cdot y \cdot z$	(7) $5 \cdot 4 \cdot 3 \cdot 6$	(11) $8 \cdot 7 \cdot 6 \cdot 5$
(4) $7x \cdot 7y \cdot z$	(8) $5 \cdot 3 \cdot 5 \cdot 4 \cdot 5 \cdot 6$	(12) $8 \cdot 7 \cdot 8 \cdot 6 \cdot 5$.

10. Compare also the following:

$$5a \times b \times c \times d, \quad 5(a \times b \times c \times d), \quad 5a \times 5b \times 5c \times 5d.$$

11. State the law of signs for multiplication and interpret the meaning of positive and negative multipliers and multiplicands (see pp. 77–85).

Exercise XXI

Simplify the following products:

1. $(+17) \times (+25) \times (-8)$
2. $(-a^2) \times (+a^4) \times (-a^3)$
3. $(-a)^2 \times (-a)^3 \times (-a)^4$
4. $(2mt^2)(-3m^2t)(-5m^3)$
5. $(-2\frac{1}{2}x^2y)(-5\frac{1}{2}x^2y^3z)(+16x^3yz^4)$
6. $6a^2bc^2 \times 4a^2b^2c \times 3a^2b^2c^2 \times 5ab^2c^2$
7. $(-5\frac{1}{3}p^2q^3)(+5\frac{1}{4}q^2r^3)(-1\frac{1}{7}r^2p^3)$
8. $(+8 \cdot 2a^2mb)(-3 \cdot 56amn^2b^2)(+4 \cdot 3m^2n^2b)$
9. $(-3xpq^2r)(-2\frac{1}{3}xp^2qr)(-4zpqr^2)$
10. $(4\frac{1}{2}t^3s^2u^5)(-2\frac{2}{3}t^2s^5u^{12})(-\frac{1}{12}t^{10}s^7u^8)$
11. $\dfrac{12x^2y^3z^5}{5} \times \dfrac{-35x^5y^2z^4}{4} \times \dfrac{x^3y^5z^2}{7}$
12. $\dfrac{-5a^m}{3} \times \dfrac{-6a^3}{25}$
13. $\dfrac{+7t^x}{24} \times \dfrac{-8t^5}{14}$
14. $\dfrac{-as^k}{5} \times \dfrac{10s^l}{2}$
15. $(a+b)^3 \times (a+b)^5$

16. $(r-s)^n \times (r-s)^3$
17. $(l^2-f^2)^k \times (l^2-f^2)^h$
18. $(x^2+y^2)^3 \times (x^2+y^2)^5 \times (x^2+y^2)$
19. $3(x+y)^5(x-y)^4 \times 4(x+y)^3(x-y)^6$
20. $\{-(a-b)\}^3 \times \{+(a-b)\}^4 \times \{-(a-b)\}^5$
21. $\dfrac{15x^5y^7(x+y)^8}{14} \times \dfrac{-21x^4y^5(x+y)^6}{30}$
22. $(a^5)^2 \times (-a^3)^4 \times (-a^2)^5$
23. $(3p^2)^4 \times (-p^3)^5 \times (-5p^4)^2$
24. $(2\frac{1}{2}x^3)^2 \times (-\frac{2}{5}x^5)^2 \times (3x^7)^2$.

Division of Monomials

122. We have found the area of a rectangle when both dimensions are known. We shall now find one dimension if the other dimension and the area are known.

1. Find the altitude of a rectangle of area 144 sq. ft., the base being 9 ft.; 16 ft.; 12 ft.; 8 ft.; 72 feet.

2. Denoting the area of a rectangle by A, the base by b, the altitude by a, find the second dimension of the following rectangles:

(1) $A=52t^2$ sq. ft.; $b=26t$ ft.
(2) $A=x^5$ sq. in.; $b=x^2$ in.
(3) $A=4\frac{1}{2}x^4y^3z^4$ sq. m.; $a=3x^2y^2z^2$ m.
(4) $A=51.6d^5f^2c^3$; $a=2.4d^3fc^2$
(5) $A=99p^4x^7l^6$; $b=11l^4x^2p^3$
(6) $A=91l^5k^5mc^5$; $b=13lk^2mc^2$
(7) $A=v^5u^6c^7$; $a=v^2u^2c^2$
(8) $A=3.5h^6x^5z^4$; $b=5h^2x^3z^4$
(9) $A=1.96z^3x^5p^6$; $b=1.4z^2p^2x^2$
(10) $A=8\frac{1}{3}x^5y^6z^7$; $a=1\frac{2}{3}x^3y^4z^5$.

3. State a short way of dividing powers of the same base.

4. State the law of signs for division (see p. 86).

Exercise XXII

Reduce the following fractions to lowest terms:

1. $\dfrac{-25a^5b^6c^7}{5a^3b^4c^5}$

2. $\dfrac{-1.69x^4y^5z^6w^5a^8}{-1.3a^4w^2x^2y^3z^4}$

3. $\dfrac{+12\frac{1}{2}t^2u^5s^4}{-2\frac{1}{2}ts^4u^3}$

4. $\dfrac{12(-a)^4b^5(-c)^6}{3(-a)^2b^3(-c)^4}$

5. $\dfrac{-28p^2q^5r^3}{5\frac{1}{4}q^2r^3}$

6. $\dfrac{420x^{29}y^{15}z^{35}}{-12x^{18}y^7z^{17}}$

7. $\dfrac{56a^xb^{10}c^9}{14a^2b^7c^6}$

8. $\dfrac{-5.85x^{2a}y^{2b}z^{2c}}{-4.5x^ay^bz^c}$

9. $\dfrac{-3\frac{5}{7}p^{3n}q^{4n}r^{9n}}{6\frac{1}{2}p^nq^{3n}r^{5n}}$

10. $\dfrac{14(a+b)^3}{-7(a+b)^2}$

11. $\dfrac{-94(x^2-y^2)^5}{-2(x^2-y^2)^3}$

12. $\dfrac{-3.43(a^2+2ab+b^2)^8}{49(a^2+2ab+b^2)^4}$

13. $\dfrac{7\frac{1}{2}\{-(p-q)\}^5}{-2\frac{1}{2}\{-(p-q)\}^2}$

14. $\dfrac{5\frac{2}{11}x^5y^4(a^2+b^2)^7}{2\frac{1}{22}x^2y^3(a^2+b^2)^5}$

15. $\dfrac{-5\frac{1}{7}(a+b)^{3x}(a-b)^{4x}}{1\frac{1}{5}(a+b)^x(a-b)^{2x}}$

16. $\dfrac{-144x^{5a}y^{7b}(x+y)^{a+b}}{-18x^{3a}y^{4b}(x+y)^b}$

17. $\dfrac{-14\frac{2}{7}p^aq^br^c(p+q-r)^{a+b+c}}{2\frac{8}{21}p^bq^cr^a(p+q-r)^a}$

18. $\dfrac{35x^4y^3z^6}{5x^6y^5z^4}$

19. $\dfrac{-1.25a^2b^5c^{15}}{-25a^5b^7c^9}$

20. $\dfrac{4\frac{2}{5}(x+y)^3(x-y)^4}{-\frac{10}{11}(x+y)^5(x-y)^7}$.

Multiplication and Division by Means of Exponents

123. Make tables of the first twelve powers of 2, 3, and 5; thus:

$2=2^1$	$3=3^1$	$5=5^1$
$4=2^2$	$9=3^2$	$25=5^2$
$8=2^3$	$27=3^3$	$125=5^3$
$16=2^4$	$81=3^4$	$625=5^4$
$32=2^5$	$243=3^5$	$3{,}125=5^5$

$64=2^6$	$729=3^6$	$15{,}625=5^6$
$128=2^7$	$2{,}187=3^7$	$78{,}125=5^7$
$256=2^8$	$6{,}561=3^8$	$390{,}625=5^8$
$512=2^9$	$19{,}683=3^9$	$1{,}953{,}125=5^9$
$1{,}024=2^{10}$	$59{,}049=3^{10}$	$9{,}765{,}625=5^{10}$
$2{,}048=2^{11}$	$177{,}147=3^{11}$	$48{,}828{,}125=5^{11}$
$4{,}096=2^{12}$	$531{,}441=3^{12}$	$244{,}140{,}625=5^{12}$

1. Using the tables, write as powers of 2, 3, or 5:

16	6,561	177,147
2,187	243	729
1,024	128	15,625
390,625	48,828,125	

2. Using the above tables, many large multiplications and divisions may be simplified, as will be seen from the following examples:

(1) $32\times64=2^5\times2^6=2^{11}=2{,}048$
(2) $15{,}625\times625=5^6\times5^4=5^{10}=9{,}765{,}625$
(3) $(729)^2=(3^6)^2=3^{12}=531{,}441$
(4) $4{,}096\div16=2^{12}\div2^4=2^8=256$
(5) $(625)^3=(5^4)^3=5^{12}=244{,}140{,}625$.

Exercise XXIII

Carry out the following multiplications and divisions, using the tables of powers of 2, 3, and 5:

1. $6{,}561\times81$
2. $78{,}125\times625$
3. $512\div16$
4. 729×243
5. $2{,}048\div128$
6. $177{,}147\div729$
7. $390{,}625\div3{,}125$
8. $244{,}140{,}625\div78{,}125$
9. $(243)^2$
10. $(15{,}625)^2$
11. $4{,}096\div64$
12. $390{,}625\times625$.

For the following problems the above tables have to be extended to the 25th powers:

13. $\dfrac{282{,}429{,}536{,}481}{3{,}486{,}784{,}401}$

14. $15{,}625 \times 78{,}125 \times 9{,}765{,}625$

15. 256^3

16. $\dfrac{847{,}288{,}609{,}443 \times 531{,}441}{31{,}381{,}059{,}609 \times 14{,}348{,}907}$

17. $\dfrac{30{,}517{,}578{,}125 \times 3{,}814{,}697{,}265{,}625}{10{,}670{,}928{,}955{,}078{,}125}$

18. $\dfrac{16{,}777{,}216 \times 1{,}024 \times 262{,}144}{2{,}097{,}152 \times 8{,}192}$.

Multiplication and Division of a Polynomial by a Monomial

124. What is the total area of four adjacent flower-beds, whose length is 10 ft. each, and whose widths are 7, 2, 5, and 3 ft. respectively?

The total area may be expressed in either of the two following ways:

$$10(7+2+5+3) \quad \text{or} \quad 10\times7+10\times2+10\times5+10\times3$$

Thence:

$$10(7+2+5+3)=10\times7+10\times2+10\times5+10\times3.$$

The area may thus be expressed as the product of a polynomial by a monomial or as one polynomial.

1. Represent in two different ways the total area of 7 adjacent rectangles having the same base b and having lengths 5, a, x, $7\frac{1}{2}$, $4\frac{1}{3}$, 9.5, and y^2 respectively. Express by an equation the equality of these two representations.

2. Represent in two different ways the combined area of 4 rectangles of length a^2 and of bases a^3, $3ab^2$, $3a^2b$, b^3 respectively. Express the equality of the two representations.

3. How may the product of a polynomial by a monomial be reduced to a polynomial? What do you notice concerning the terms of the resulting polynomial?

Exercise XXIV

Reduce to one polynomial:

1. $5x(x^2-3xy+5y^2)$
2. $6\frac{2}{3}ab(3a^2-6ab+12b^2)$
3. $-5.7x^3(\frac{12}{19}abx-1\frac{4}{19}aby-2\frac{2}{19}xyz+\frac{7}{19}byz)$
4. $-ap^2m^3(+4a^2p^3m-5a^3pm^2-7\frac{1}{2}ap^3m^2-4.5a^3p^2m)$
5. $3a(4a-5b)+5b(6a-3b)$
6. $5a^2(2\frac{1}{5}a^2-4\frac{1}{2}ab-3.5b^2)-4b^2(3\frac{1}{3}a^2+2ab-2\frac{3}{8}b^2)$
7. $4x^2(3x-2y)-2xy(3x-2y)+6y^2(3x-2y)$.
8. $2(3a^2-4\frac{1}{2}ab+7b^2)-3(4a^2+5\frac{2}{3}ab-8\frac{1}{2}b^2)+$
$4\frac{1}{2}(-2\frac{1}{5}a^2-6ab-5b^2)$
9. $5x\{4a-2(3a-4b)+5(4a-3b)\}$
10. $2a\{5(4a-7b-3c)-6(5a+4b-8c)\}$
11. $-4x[2x^2+3x\{4(x-1)-5(x-2)\}]$
12. $+5y^3-[3y^3-2y\{4y(y+3)-5y(2y+6)\}]$.

Solve the following equations:

13. $4(x-6)-3(x+3)=15$
14. $3\frac{1}{2}(3x-8)-4\frac{1}{2}(3x+3\frac{7}{9})=0$
15. $-5x(4x+2y+6)+3x(7x+6y-9)-(x^2+8xy-7x)$
$=100$
16. $4a[3a-(5a+2)]-2a[-3a-(a-4)]-4(-3a-5)=0$.

125. To factor a number is to find the numbers which, multiplied, give the number to be factored.

1. The total area of three adjacent lots is $x^4+3x^3y+4x^2y^2$ sq. m., each term representing the area of a lot. The lots all have the same length. What may their dimensions be?

2. Sketch a rectangle whose area is:

(1) $4x^2-3xy$
(2) $5x^3-10x^2y+15xy^2$
(3) $14a^2b^3c^2-21a^3b^2c^2+35a^2b^2c^3$
(4) $3m^5-12m^3n+6mn^4$
(5) $15x^4-10x^3+5x^2$.

3. How can some polynomials be reduced to the product of a polynomial by a monomial? Can every polynomial be factored in this way?

Exercise XXV

Write the following polynomials as the product of a polynomial by a monomial:

1. $5a-10b$
2. $17x^2-289x^3$
3. $16x^2-2abx$
4. $ax+ay-az$
5. $14x^2y^2z^2-7x^3y^3z^2+8xy^2z^2$
6. $60m^2n^3r^2-45m^3n^2r^3+90m^4n^2r^3+90m^4n^3r^2$
7. $4.5a^2b^3c^4-1.2a^3b^2c^4-2.7a^4b^2c^3+9a^3b^4c^2$
8. $4a^2x^3-12a^3x^4-20a^4x^3$
9. $ap^3+3p^2x-4pq+15apxq$
10. $x^3-pqx^2+p^2q^2x-pq^3x^3$.

126. When one of the factors of a number is known, a second factor is found by dividing the known factor into the given number.

1. The area of a garden is $15abc+27a^2c-21b^2c$. The length is $3c$. Find the width.

2. Find the missing dimension of the following rectangles:

	AREA	BASE	ALTITUDE
(1)	s^4+s^7	s^2	—
(2)	$8x^2y^2-10xy^3-4x^4$	—	$2x$
(3)	$pk+lk-rk+tk$	—	k
(4)	$mu^2+mv^3+mw^4-mz$	m	—
(5)	$as+bs-cs+ds$	s	—

3. How may the quotient of a polynomial by a monomial frequently be reduced to a polynomial? Is this always possible?

Exercise XXVI

Reduce:

1. $\dfrac{20a^3b-15a^2b^2+30ab^3}{5ab}$

2. $\dfrac{10x^2y-15x^3y^2+5x^3y}{-5xy}$

3. $\dfrac{36x^2y^4-42x^5y^6z}{-6x^2y}$

4. $\dfrac{35ab^8c^3-42a^3b^4c^3-49a^8b^4c^3+21a^3b^5c^7}{7ab^3c^3}$

5. $\dfrac{12am^2n^3-16bm^3n^2+40abm^2n^2-28a^2m^2n^2}{-4mn}$

6. $\dfrac{-15p^{15}q^{10}-12p^{27}q^{14}+9a^3p^{14}q^{16}-27abcp^5q^{10}}{3p^2q^3}$

7. $\dfrac{4abc(3abc-5a^2b^2c^2-7ab^2c+6ab^4c)}{-2a^2bc^2}$

8. $\dfrac{2x(4yz-6x^2y^2)-8y^2(5xz^2-1\frac{1}{2}x^3)}{4xyz}$

9. $\dfrac{15a^3b^4c^5-9a^4b^3c^2-45a^2b^3c^5+21a^5b^4c^3-3a^4b^5c^4}{3a^2b^3c^2}$

10. $\dfrac{21abdpqs-35abcpql-42acdpts}{-7ap}$.

Exercise XXVII

Reduce to a polynomial:

1. $4\frac{1}{2}abc(\frac{2}{3}a^2-\frac{4}{5}b^2+\frac{1}{3}c^2)$
2. $3\frac{1}{2}x^2y^3(x^3y^4-4x^4y^5+6x^2y^7-9x^5y^4)$
3. $-5a[4abc(3a^2b-5ab^2+3a^2c-6ac^2)-3a^2bc^2(4a-5b-6c).]$

Reduce to the product of a polynomial by a monomial or by a binomial:

4. $25a^3b^4c^7d^8-15a^3b^5c^6d^2+30ab^2c^3d^9$
5. $21x^5y^9z^{15}-14x^6yz^{14}+35xy^{10}z^{12}-42x^{10}y^{15}z$
6. $4a(x-y)-3b(x-y)+4c(x-y)$.

Reduce to a polynomial:

7. $\dfrac{15x^{2a}y^{2b}z^{2c}-18x^{3a}y^{3b}z^{3c}-21x^{4a}y^{4b}z^{4c}}{3x^{a}y^{b}z^{c}}$

8. $\dfrac{15(x+y)^5-25(x+y)^8-35(x+y)^3}{-5(x+y)^3}$

9. $\dfrac{[21pm(r^2+s^2)(r+s)-42mp(r^2+s^2)^2-7m](r^2+s^2)(r+s)}{-7m(r^2+s^2)(r+s)}.$

Solve the following equations for c:

10. $4c(x-3y+4z-2)-4c(4z-5)+3c(4y-2)-c(4x-1)=35$

11. $5c[4c^3-3c(c+2)]-5\{c^3-\overline{6c^2-c}\}=20$

12. $3a(c-4b)-2b(4c-6a)-4c(a-2b)=10a.$

Multiplication of Polynomials

127. When the dimensions of a rectangle are polynomials, the area can be found, by separating the rectangle into other rectangles whose dimensions are simpler numbers.

1. Draw a rectangle of length $x+y$ and of width $a+b$. Determine the area of the rectangle and write your answer in the form of a polynomial.

2. Express the area of the rectangle of Fig. 196 (1) as the sum of two rectangles; (2) as the sum of four rectangles.

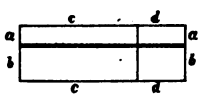


FIG. 196

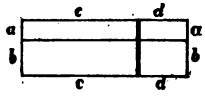


FIG. 197

3. Express the area of the rectangle of Fig. 197 (1) as the sum of two rectangles; (2) as the sum of four rectangles.

4. Write the area of the rectangles, whose dimensions are given below, (1) as the sum of two or more rectangles, (2) in the form of a polynomial. Simplify the polynomial:

	LENGTH	WIDTH
(1)	$3+4$	$6+2$
(2)	$5+a$	$a+3$
(3)	$a+6$	$a-2$
(4)	$a+b$	$b+c$
(5)	$x+y$	$y+z$
(6)	a^2+b^2+2ab	$a+b$
(7)	$x^2+2xy+y^2$	$x-y$
(8)	$a-b$	$a^2-2ab+b^2$
(9)	$x+y$	$x^2-2xy+y^2$
(10)	$5+x$	$25+10x+x^2$.

5. From the preceding problems make a rule for expressing the product of two polynomials in the form of a single polynomial.

128. The rectangles of problems 2, 3, and 4, §**127**, are first expressed as the sum of two or more rectangles; for example, for the rectangle of problem 2:

$$(a+b)(c+d)=a(c+d)+b(c+d).$$

The final product of the two polynomials, which represent the dimensions of the rectangle, is thus made up of the sum of products of one of the polynomials by the several terms of the other polynomial. These separate products are called the *partial products*. They represent what each term of one of the polynomials in connection with the entire second polynomial contributes to the final product.

1. State the partial products for each of the parts of problem 4, also stating how each of these is obtained.

2. Most frequently the partial products are written, one underneath the other, so as to bring all similar terms in the same column. The work is then arranged as follows:

$$\begin{array}{lll}
 & & a^2+ab+b^2 \\
 & & a\ \ +b \\
\hline
\text{First partial product:} & a(a^2+ab+b^2): & a^3+\ a^2b+\ ab^2 \\
\text{Second partial product:} & b(a^2+ab+b^2): & \qquad\ a^2b+\ ab^2+b^3 \\
\hline
 & & a^3+2a^2b+2ab^2+b^3
\end{array}$$

Exercise XXVIII

Simplify the following products:

1. $(ax+by)(ax-by)$
2. $(a^3+ab^2+b^3)(a+b)$
3. $(p^2+2p+1)(p-3)$
4. $(2\frac{1}{2}a^2-3\frac{1}{3}b^3)(12a^3-6b^2)$
5. $(4.6abc+1.2ab^2)(4a^2c-5a^2b)$
6. $(a^2+ab+b^2)(a-b)(a+b)$
7. $(4x^2+2xy+y^2)(2x+y)(2x-y)$
8. $(p^2+pr+r^2)(p^2-pr+r^2)$
9. $(9k^2-3kt+t^2)(9k^2+3kt+t^2)$
10. $(2\frac{1}{3}m^2-4\frac{1}{5}ms+3\frac{1}{2}s^2)(\frac{30}{7}m-\frac{60}{7}s)$
11. $(3ab-5bc)(2a-3b+5c)+(4ac-6ab)(3c^2-4b^2+2a^2)$
12. $(9x^2-6xy+4y^2)(9x^2+6xy+y^2)$
13. $(5a+2b)^3-(5a-2b)^3$
14. $(2f-3h)^2-(2f+3h)^2+(2f-3h)(2f+3h)$.

129. The squares of binomials can be deduced readily from the binomials without actually performing the multiplications.

1. Express in the form of a trinomial by means of multiplication:

$(m-x)^2$	$(p+k)^2$
$(a+b)^2$	$(x-y)^2$
$(p-k)^2$	$(m+x)^2$.

Notice how the terms of the binomial that is squared are used to form the terms of the final result.

2. Write down at once in the form of a trinomial:

$(x+p)^2$	$(k-r)^2$	$(r-y)^2$
$(v+a)^2$	$(r+y)^2$	$(h-p)^2$
$(x-p)^2$	$(k+n)^2$	$(g-f)^2$.

3. From the preceding problems make a rule for expressing the square of a binomial.

Exercise XXIX

Write in the form of a trinomial, doing all you can mentally:

1. $(x+5)^2$
2. $(y-7)^2$
3. $(4a-1)^2$
4. $(\frac{1}{3}x+\frac{1}{2}y)^2$
5. $(.4a-.3t)^2$
6. $(3px-4qy)^2$
7. $(.6xyz+1)^2$
8. $[(a+b)+3]^2$
9. $[(a-b)+c]^2$
10. $[(x-y)-\frac{4}{5}]^2$
11. $[(k+l)-1.5s]^2$
12. $[(3a-4b)-2c]^2$
13. $[(ax+by)-cz]^2$
14. $[(5pq-3kl)-4pl]^2$
15. $[(2\frac{1}{3}ab^2c-\frac{6}{7}abc^2)+7a^2bc]^2$
16. $[(a+b)+(c-d)]^2$
17. $(2a+3b)^2-(3a-2b)^2+(4a-3b)^2$
18. $[(4t-5x)+(5s-4y)]^2$.

130. The product of the sum of two numbers by the difference of the same numbers can be readily obtained without actually multiplying.

1. Express as a binomial by multiplying:

$(a+x)(a-x)$ $\quad$ $(p+s)(p-s)$
$(r+4)(r-4)$ $\quad$ $(y+b)(y-b)$
$(t+q)(t-q)$ $\quad$ $(15-z)(15+z)$.

2. Write in the form of a binomial without multiplying:

$(k+h)(k-h)$ $\quad$ $(4a-3b)(4a+3b)$
$(2s-13)(2s+13)$ $\quad$ $(\frac{1}{2}p-\frac{1}{3}r)(\frac{1}{2}p+\frac{1}{3}r)$
$(m+v)(m-v)$ $\quad$ $(.3xl+.2yk)(.3xl-.2yk)$
$(t-u)(t+u)$ $\quad$ $(abc-xyz)(abc+xyz)$.

3. From the preceding exercises make a rule for expressing as a binomial the product of the sum of two numbers by the difference of the same numbers.

Exercise XXX

Express as binomials and perform the operations indicated in the resulting binomials:

1. $(2a+3b)(2a-3b)$
2. $(3x-4z)(3x+4z)$
3. $(5p+3s)(5p-3s)$
4. $(\frac{1}{2}r+\frac{1}{3}t)(\frac{1}{2}r-\frac{1}{3}t)$
5. $(\frac{2}{5}xy-\frac{2}{3}xz)(\frac{2}{3}xz+\frac{2}{5}xy)$
6. $(4abc-3)(4abc+3)$
7. $(1+5p^2q^2)(1-5p^2q^2)$
8. $(1-a)(1+a)(1+a^2)$
9. $(u-v)(u+v)(u^2+v^2)$
10. $[(a+b)+c][(a+b)-c]$
11. $[(e-b)-h][(e-b)+h]$
12. $[(m+n)-t^2][(m+n)+t^2]$
13. $[(2a-b)-3s][(2a-b)+3s]$
14. $(k-l)(k+l)(k^2+l^2)$
15. $(g-f)(g+f)(g^2+f^2)(g^4+f^4)$
16. $(1-p)(1+p)(1+p^2)(1+p^4)$
17. $[(3x^4-2y^3)-xy][(3x^4-2y^3)+xy]$
18. $[(\frac{2}{3}x^5+\frac{4}{5}y^7)-\frac{1}{2}x^{10}y^2][(\frac{2}{3}x^5+\frac{4}{5}y^7)+\frac{1}{2}x^{10}y^2]$
19. $(a^m+b^4)(a^m-b^4)$
20. $(x^a-y^s)(x^a+y^s)$
21. $(2m^{2t}-3s^{3v})(2m^{2t}+3s^{3v})$
22. $[(a+b)-(c+d)][(a+b)+(c+d)]$
23. $[(x^2-y^2)-(x^3+y^3)][(x^2-y^2)+(x^3+y^3)]$
24. $[(1.2ab+\frac{1}{3}bc)-(.4ac-2b^2)][(1.2ab+\frac{1}{3}bc)+(.4ac-2b^2)]$
25. $[(\frac{1}{2}x^2-\frac{2}{3}y^2)+(\frac{3}{2}x^2+\frac{1}{3}y^2)][(\frac{1}{2}x^2-\frac{2}{3}y^2)-(\frac{3}{2}x^2+\frac{1}{3}y^2)]$.

Exercise XXXI

Multiply as indicated:

1. $2a^2(3a^3-4b^3)-3a^3(4a^2-2b^2)$
2. $(\frac{1}{3}pt-\frac{1}{2}ts)(\frac{3}{4}pt-\frac{2}{3}ts)$

3. $(a+b+c)^2$
4. $(a+b-2c)^2$
5. $(-2a-3b+c)^2$
6. $(a^2+b^2-3c^3)^2$
7. $(2a^2+b^2+3c^2)(2a^2+b^2-3c^2)$
8. $(.5x-.4y-.3z)(10x-20y+30z)$
9. $4\frac{1}{2}mn(4m^2-6mn+9n^2)-3mn^2(13\frac{1}{2}n-9m)$
10. $(3y^2-45xy+6x^2)(3y^2-5x^2)$
11. $(ax+by-cz)(ax+by+cz)$
12. $(ax+by-cz)^2$
13. $(ax-by-cz)^2$
14. $(ax-by+cz)^2$
15. $(\frac{1}{3}ms-\frac{1}{4}st-\frac{2}{3}tm)(6m-12s+18t)$
16. $(-1.4l^2-2.5ml+.9m^2)(.2l^2-lm-.1m^2)$
17. $(ax-by+cz)^2-(ax+by+cz)^2$
18. $(a+b)^3$
19. $(a-b)^3$
20. $(2x-y)^3$
21. $(x+2y)^3$
22. $(3x+4y)^3-(3x-4y)^3$
23. $(\frac{1}{2}m^2-\frac{1}{3}v^2)^3+(\frac{1}{2}m^2+\frac{1}{3}v^2)^3$
24. $(.5x-.6y)^2-(.5x+.6y)^2-(.5x+.6y)(.5x-.6y)$
25. $(2p^2+q^2)^3-(2p^3-q^3)^2-4p^4(p-2q)^2$.

Division of Polynomials

131. In the division of polynomials, all the preceding operations find application. It is therefore a subject by means of which the foregoing work may be reviewed.

1. Write the product of $x+y$ by $a+b+c$ in the form of a polynomial.

What is the contribution of a and $x+y$ to the product? Of b and $x+y$? Of c and $x+y$?

From what terms of the multiplier and the multiplicand is the term ax of the product obtained?

2. What is to be found by dividing $ax+ay+bx+by+cx+cy$ by $x+y$?

How is the first term of the quotient found?

How is the total contribution of the divisor and the first term of the quotient obtained?

Write down the remainder of the dividend, which is contributed by the divisor and the remaining terms of the quotient, as yet unknown.

3. Determine the 2d term of the quotient from the remainder of the dividend by the same method as was used in problem 2 to determine the first term of the quotient. Then proceed as in problem 2. Repeat this process, until all the terms of the dividend have been used.

The following arrangement of the work is convenient:

$$
\begin{array}{lr}
 & a+b+c \\
 & x+y\,\overline{)\,ax+ay+bx+by+cx+cy} \\
\text{First partial product: } a(x+y) & ax+ay \\
\hline
 & bx+by+cx+cy \\
\text{Second partial product: } b(x+y) & bx+by \\
\hline
 & cx+cy \\
\text{Third partial product: } c(x+y) & cx+cy \\
\hline
 & 0
\end{array}
$$

4. Compare the partial products of problem 1 at the beginning of this section with the partial products in the division.

5. Divide $a^3+3a^2b+3ab^2+b^3$ by $a+b$. What is the first term of the quotient? How is the 1st partial product found? What is the 2d term of the quotient? How is the 2d partial product found? etc.

Exercise XXXII

Divide, and test by multiplying:*

1. $9t^2+24st+16s^2$ by $3t+4s$
2. $1.2x^2+.5xy-2.8y^2$ by $4x+7y$
3. $6k^3-31k^2t+47kt^2-42t^3$ by $2k-7t$
*4. $10x^3-5xy^2-54y^3-4x^2y$ by $x-2y$
5. $27a^3-54a^2b+36ab^2-8b^3$ by $3a-2b$
6. $27a^3-54a^2b+36ab^2-8b^3$ by $9a^2-12ab+4b^2$
7. x^3-y^3 by $x-y$
8. $27t^3-64s^3$ by $3t-4s$
9. $64a^6-b^6$ by $8a^3-b^3$
*10. $3y^4+15x^3y-9xy^3-25x^4+10x^2y^2$ by $5x^2-3y^2$
11. $8x^3-y^3$ by $4x^2+2xy+y^2$
12. $64a^6-b^6$ by $4a^2-b^2$
13. $.008s^3t^3+v^3$ by $.2st+v$
14. $a^3+a^2b+ab^2+ac^2+bc^2+b^3$ by $a+b$
15. $64a^6-b^6$ by $16a^4+4a^2b^2+b^4$
16. $4x^6+12x^3y^3+9y^6$ by $2x^3+3y^3$
17. $.008s^3t^3+v^3$ by $.04s^2t^2-.2stv+v^2$
18. $16x^4-25r^4$ by $4x^2-5r^2$
19. $a^3+a^2b+ab^2+ac^2+bc^2+b^3$ by $a^2+b^2+c^2$
20. $9t^6-12t^3s^3+4s^6$ by $3t^3-2s^3$
21. $27m^6-8n^6$ by $9m^4+6m^2n^2+4n^4$
22. $16x^4-25r^4$ by $4x^2+5r^2$
23. $27m^6+8n^6$ by $3m^2+2n^2$
24. $64u^6+14.4u^4v^2+1.08u^2v^4+.027v^6$ by $4u^2+.3v$
25. $27m^6-8n^6$ by $3m^2-2n^2$.

* Before beginning to divide arrange the dividend and the divisor according to powers of the same number. Arrange the successive remainders in the same order. Although the work can sometimes be done just as well without paying any attention to the arrangement, it will be found convenient to follow the above rule.

Summary

Terms which have a *common factor* are called **similar with respect to that factor.**

Terms which have *no common factor* are called **dissimilar**

Polynomials consisting of *similar terms* may be simplified by adding the coefficients of the common factors in the similar terms.

Polynomials, no two of whose terms are similar, cannot be simplified.

Subtraction of monomials and of polynomials may always be *reduced* to **addition.**

If a polynomial is in parentheses preceded by a minus sign, the subtraction indicated may be changed into addition by changing the sign of each term in the parentheses.

The **value** of a **product** remains unchanged, when the **order** of the factors is changed.

A number, which is expressed as the product of two or more factors, is multiplied by multiplying **one factor.**

Powers of the same base are *multiplied* by adding the exponents of the powers.

Two powers of the same base are *divided* by *subtracting* the exponent of the divisor from the exponent of the dividend.

Multiplication and **division** of arithmetical numbers may be simplified by the use of *exponents.*

To multiply a polynomial by a monomial, multiply *each term* of the polynomial by the monomial and *add* the separate products algebraically.

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, then add the separate quotients algebraically.

The product of two polynomials is obtained by *multiplying one of the polynomials* by *every term* of the other polynomial, and adding the partial products algebraically.

The process of dividing one polynomial by another is the reverse, step by step, of the process of multiplying polynomials.

CHAPTER IX

PRACTICE IN ALGEBRAIC LANGUAGE. GENERAL ARITHMETIC

Translating Verbal into Symbolic Language

132. The following exercises give practice in translating number expressions and relations from verbal into symbolical language and the reverse. Answer all you can orally.

1. Give the three consecutive integers that begin with 16; end with 16.

2. Express the three consecutive integers that begin with n; end with n.

3. Express the three consecutive even numbers that begin with $2x$; end with $2x$.

4. Express the three consecutive odd numbers that begin with $n+3$; end with $n+3$.

5. Express the sum of the three integers that begin with x and differ by 3.

6. Express the sum of the three consecutive integers, the second integer being n.

7. Express in symbols that the sum of three consecutive integers equals 21. Find the three integers.

8. Express in symbols that the product of two consecutive integers equals 72.

9. Express in symbols that the sum of three consecutive integers, differing by 3, equals 27. Find the three integers.

10. The tens' digit of a certain number is 4 and the units' digit is 3. Indicate in symbols the number of units in the tens.

11. The tens' digit of a number is t and the units' digit is u. Indicate the number of units in the number.

12. The tens' digit of a number is twice the units' digit. Indicate the number in symbols.

13. The present age, in years, of a person is denoted by x. Indicate in symbols the following:

(1) The person's age 12 years ago
(2) His age 12 years hence
(3) His age when twice as old
(4) Four times his age 8 years ago
(5) Six times his age 3 years hence
(6) One-eighth of his age 15 years ago
(7) Three times his age less twice his age 6 years ago
(8) 60 decreased by his age
(9) His age decreased by 60
(10) His age increased by one-half his age
(11) His age diminished by 3 times his age 20 years ago
(12) The ratio of his present age to his age 14 years hence.

14. Letting a certain number be denoted by $10t+u$, what will denote the number formed by reversing the digits?

15. If x denotes the number of minute-spaces the tip of the hour hand moves over in a given time, what will denote the number of minute-spaces the tip of the minute hand moves over in the same time?

16. If x minutes denote the space moved by the minute hand, what denotes the space moved by the hour hand in the same time?

17. Express in symbols that a number whose units' digit is 3 less than the tens' digit equals 27 more than the number made by writing the digits in reverse order.

18. Express that one-eighth of the double of n increased by 25, equals one-half of n.

19. Write in symbols an expression for double a number, increased by 3 times the sum of the number and 4.

20. Write an expression for double a number, decreased by 3 times the difference between the number and 4.

In the phrases "the difference between 8 and 4," "the difference of 8 and 4" and the like, it is understood that the first-mentioned number (the 8) is the minuend; thus, $8-4$, not $4-8$.

21. Write the double of a number, decreased by 3 times the difference of the number and 2.

22. Write in symbols 4 times the difference of a number and 3, decreased by 3 times the difference of the number and 1.

23. Show, in symbols, that the double of a number, increased by 3 times the difference of the number and 4, equals 13.

24. Find the value of the following for $x=12$:

$2x+3(x+4)$	$2x-3(x+4)$
$2x+3(x-4)$	$2x-3(x-4)$
$3(x-4)-2x$	$4(x-5)-3(x-6)$.

25. Translate the six expressions of problem 24 into words.

26. Write in symbols the following:

(1) One-third the sum of r and 4
(2) The sum of r and 4, divided by 3
(3) One-half the difference of twice s and 11
(4) The difference of twice s and 11, divided by 2
(5) Two-thirds of the difference of $8y$ and 20
(6) Two times the difference of $8y$ and 20, divided by 3
(7) One-fourth of 5 times the sum of $2x$ and 9
(8) Five-fourths of the sum of $2x$ and 9.

27. Show, in symbols, a number whose digits are x and 4; x and y; y and x.

28. Show, in symbols, a three-digit number whose digits are a, b, and c; c, a, and b; x, z, and y; y, z, and x.

The Operations of Arithmetic Symbolized

133. The work of algebra requires that the operations used to obtain results be *indicated* more frequently and more fully than they are in arithmetic.

1. Show by an equation that 12 is the sum of 7 and 5.

2. Indicate that 45 is the sum of 4 tens and 5.

3. Indicate that 68 is the sum of 6 tens and 8.

4. Indicate the sum of 50 and 7; 48 and 9; 18 and x.

5. What is the sum of 8 and x? Of 15 and y? Of x and 16? Of x and y?

6. Show by an equation that s is the sum of a, a, a, a, and 10.

7. The minuend is 18 and the subtrahend 7. Express by an equation that the difference is 11.

8. Denote the minuend by m, the subtrahend by s, and the difference by d. Express by an equation the relation of these numbers.

EQUATION: $d=m-s$. INTERPRETATION: "difference equals minuend minus subtrahend.

9. Express by an equation that 3 times a equals $3x$.

10. The multiplicand is M, the multiplier m, and the product P. Express their relation by an equation and translate the equation into words.

11. Add s to both sides of the equation, $d=m-s$, and translate the result. Show that the result is a rule for checking, or testing, subtraction.

12. Divide both sides of $P=M\cdot m$, or $P=Mm$, (1) by m (m-minor) and interpret (translate into words) the result; (2) by M (M-major), and interpret the result.

13. Show by an equation the relation between the dividend, D (D-major), divisor, d (d-minor), and the quotient, q, when there is no remainder. Interpret the equation.

14. Show by an equation the relation of the dividend D,

divisor, d, quotient q, and the remainder r, and interpret the equation.

Express by equations the relations of the numbers in the following problems:

15. A boy has m marbles and buys b more. He then has M marbles.

16. There are b boys and g girls in a class of p pupils.

17. A boy earns c cents a day for d days. He then has s cents.

18. A rectangular flower bed is f feet long and contains s square feet. The width is w feet.

19. A bicyclist rides M miles, which is r miles more than t times b miles.

FRACTIONS

20. The sum of the fractions $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is S.

21. The difference of the fractions $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is D. $\left(\frac{n_1}{d_1}\right.$ is the minuend-fraction.)

22. The product of the fractions $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is P.

23. The quotient of the fraction $\frac{n_1}{d_1}$ divided by $\frac{n_2}{d_2}$ is Q.

MENSURATION

24. The base of a rectangle is b ft. and the altitude is a ft. The area is R sq. feet.

25. Each side of a square is s ft. and the area is A sq. feet.

26. The value of a load of corn of b bu. at s ct. per bushel is D dollars.

DECIMAL FRACTIONS

27. A decimal fraction has three units in tenths' place, and 5 units in hundredths' place, and the value is v.

NOTE. $v = \frac{3}{10} + \frac{5}{100}$.

28. A decimal fraction has t units in tenths' place and h units in hundredths' place and the value is v.

29. A decimal has a units in tenths' place and b units in hundredths' place and c units in thousandths' place. The value is v.

30. A mixed number has a_1 units in hundredths' place, a_2 units in tenths' place, a_3 in units' place, a_4 in tens' place, and a_5 in hundreds' place. The value is v.

31. A mixed number has a_1 units in hundredths' place and a_2 units in hundreds' place. The value is v.

PERCENTAGE

32. Find 6% of $210; of 520 acres; of 650 miles; of b yards.

33. Find the percentage, p, of $450 at 5%; of 375 bushels at 20%; of 1,800 men at 15%; of b acres at 25%; of b at r%.

These problems show that the law of percentage may be written thus: $p=\frac{b \cdot r}{100}$, or $p=\frac{br}{100}$.

34. Divide both sides of the equation $p=\frac{b \cdot r}{100}$ by b and interpret the resulting equation.

35. Multiply both sides of $p=\frac{b \cdot r}{100}$ by 100, then divide both sides of the resulting equation by b, and interpret the result.

36. Given the percentage, 28.80, and the base, 360, find r, the rate. Use the result of problem 35.

37. Given the percentage, 30, and the base, 725, find the rate, r.

38. Multiply both sides of $p=\frac{b \cdot r}{100}$ by 100 and divide both sides of the resulting equation by r, and interpret the result.

39. Given the percentage, 27, and the rate, 5, find the base, b.

40. Given the percentage, 31.5, and the rate, 7, find the base, b.

41. Multiply both sides of $p=\frac{b \cdot r}{100}$ by 100 and interpret.

42. The percentage is 77.4. Find the product of the base and rate.

SIMPLE INTEREST

43. Find the simple interest at 6% on \$65 for 1 year; for 2 years; for 3 years; for 5 years; for $2\frac{1}{2}$ years; for $3\frac{3}{4}$ years; for t years.

44. Find the simple interest for t years on \$80 at 4%; at 5%; at 6%; at $4\frac{1}{2}$%; at 5.6%; at r%.

45. Find the simple interest at r% for t years on \$20; on \$75; on \$135; on \$$p$.

46. Calling i the simple interest on a principal of \$$p$, at r%, for t years, show by an equation how i, p, r, and t are related.

47. Divide both sides of the equation, $i=\frac{prt}{100}$, by rt, then multiply both sides of the resulting equation by 100, and interpret.

48. Divide both sides of $i=\frac{prt}{100}$ by pr, then multiply both sides by 100, and interpret.

49. Multiply both sides of $i=\frac{prt}{100}$ by 100, then divide by pt, and interpret.

What is the product of a fraction by the denominator? Illustrate.

134. Equations, such as $d=m-s$, $P=m \cdot M$, $D=q \cdot d+r$, $\frac{n_1}{d_1}-\frac{n_2}{d_2}=\frac{n_1d_2-n_2d_1}{d_1 \cdot d_2}$, $\frac{n_1}{d_1} \div \frac{n_2}{d_2}=\frac{n_1d_2}{d_1n_2}$, $p=\frac{br}{100}$, etc., are called **formulas;** because they express laws of number in *brief form*, that is, they **formulate** laws of number.

Graphing Percentage and Interest

135. The laws of percentage may be more clearly shown graphically.

PERCENTAGE

1. What is 2% of $100? Of $200? Of $300?

2. Let a vertical side of a small square (Fig. 198), denote $2, and mark upward from the horizontal O X, on the verticals

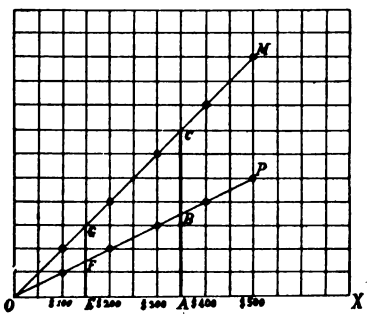


FIG. 198

through $100, $200, $300, $400, etc., the percentages at 2% of $100, $200, $300, etc., as shown. Draw a line, as O P, through the tops of the lines. What kind of line does O P seem to be?

3. Starting again on the $100-vertical, mark off similarly distances above O X to show 4% of $100, of $200, of $300, of $400, etc., to $500, and connect the upper ends with a continuous line, as O M.

4. Draw a similar line for the percentages at 3% of the same amounts.

5. Draw a similar percentage-line for 1% of the same amounts.

6. Through what point do all these percentage-lines pass? What does this mean?

7. How does the percentage-line at 3% lie with respect to the lines at 2% and at 4%?

8. Show that the line O P might be represented by

$$\text{percentage} = .02 \times \text{base, or more briefly, by}$$

$$p = .02 \times b, \text{ or } p = \tfrac{1}{50} b.$$

9. Write similar equations for the 4%-line; the 3%-line; the 1%-line.

10. Write similar equations for a 5%-line; a 10%-line; a 6%-line; and an r%-line.

11. In Fig. 198 what does the line, A B, drawn midway between the \$300-vertical and the \$400-vertical from O X up to O P, represent? What would A C represent? E F? E G?

12. How would you read from the figure the percentage at 4% for \$150? For \$250? For \$450?

13. Draw a number of equally spaced verticals upward from the horizontal and mark the lines 0%, 1%, 2%, etc., as shown. Taking \$500 as a base, measure upward from O X on the 1%-vertical, 1% of \$500, or \$5, letting a side of a small square division denote \$5. This gives the point a. On the 2%-line measure and mark off 2% of \$500; on the 3%-line, 3% of \$500, and so on. Connect a, b, c, d, etc., by a line.

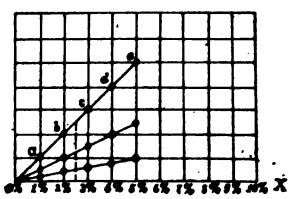


FIG. 199

14. How could one read from the drawing $2\frac{1}{2}$%, $2\frac{1}{4}$%, $4\frac{3}{4}$%, ... of \$500?

15. Taking a base of \$250 construct a \$250-percentage line.

16. Construct a \$100-percentage line.

17. Show that the \$500-percentage line might be expressed as an equation, thus:

$$\text{Percentage} = \$500 \times \text{rate},$$

$$\text{or } p = \frac{\$500r}{100},$$

$$\text{or } p = \$5r.$$

18. Write a similar equation for a $100-percentage line; a $400-percentage line.

19. All three lines go through the 0-point. What does this signify?

136. The laws of simple interest may be shown and interest-problems may be solved graphically.

1. Find the interest on $100 at 3% for 1 year; 2 years; 3 years; 4 years.

2. Call a side of a small square $3, mark a point, as *a*, to show $3 on the 1-year vertical. Mark off $6 on the 2-year vertical, as at *b*; $9 on the 3-year vertical, etc. Through the points *a*, *b*, and *c*, draw a straight line and extend it. This is the 3%-line.

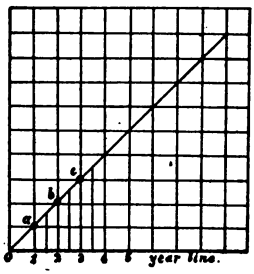


FIG. 200

3. Tell from the drawing the interest on $100 for $1\frac{1}{2}$ years at 3%; for $2\frac{1}{2}$ years at 3%; for $3\frac{1}{2}$ years at 3%.

4. Calculate the interest on $100 at 5% for 1 year; 2 years; 3 years; 4 years; etc., and draw the 5%-line.

5. From the 5%-line give the interest at 5% on $100 for $1\frac{1}{2}$ years; for $2\frac{1}{2}$ years; for $3\frac{1}{2}$ years.

6. Calculate three points and draw the 8%-line.

7. All these lines go through the zero-point, 0. What does this mean?

8. Explain how the lines show that the interest at a given rate on $100 increases proportionally to the time.

Such work as that above is called *plotting*, or *graphing*. Percentage-lines and interest-lines are called *graphs*.

137. Graphing Statistics Expressed as Percentage.

1. Taking the average price of food in the United States from 1899 to 1900 as 100, the relative prices from month to

month for 1905–6 and 1907 are given below. Graph the values for 1905–6–7 on a sheet of rectangle paper as shown for 1905. Plot the tenths by estimate.

Month	Year 1905	Year 1906	Year 1907
January	115.6	117.0	121.6
February	115.8	116.2	121.1
March	113.8	115.0	119.6
April	112.0	113.9	118.3
May	110.7	113.0	117.6
June	109.8	113.0	117.8
July	109.7	113.3	118.4
August	110.1	113.8	119.3
September	110.8	115.2	121.1
October	112.1	117.1	123.4
November	113.8	119.4	124.2
December	115.2	121.2	125.0

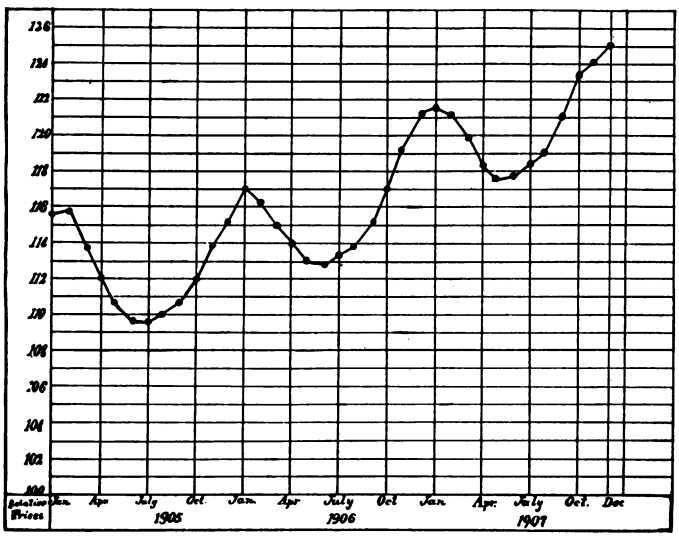


FIG. 201

2. When did the price of food reach the highest point? The lowest?

3. In what month does the price of food seem to reach a maximum? A minimum?

4. What is the general trend of prices, a rise or a fall?

5. Using 100 for the average price of food for 1890 to 1899, the relative prices for the years 1890 to 1907 are given here. Graph the values on rectangle paper.

Year	Per cent.	Year	Per cent.	Year	Per cent.
1890	102.4	1896	95.5	1902	110.9
1891	103.7	1897	96.4	1903	110.2
1892	102.0	1898	99.8	1904	111.7
1893	104.3	1899	99.6	1905	112.4
1894	99.7	1900	101.0	1906	115.7
1895	97.8	1901	105.1	1907	120.7

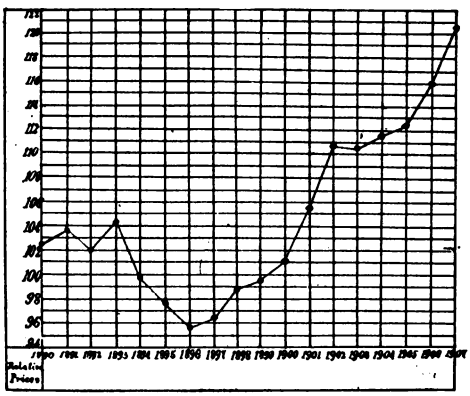


FIG. 202

6. When did the price of food reach a maximum?

7. When did the price of food reach a minimum and then rise?

138. Graphing "Sun Fast" and "Sun Slow" Data of the Almanac

Almanacs say "sun fast" and "sun slow" according as the sun, in its daily motion, reaches the meridian *ahead of* or *behind* the average time for the year. There are two causes of this difference:

I. The speed of the earth is different at different places in its orbit, and this makes the sun *appear* to move at different rates;

II. The apparent path of the sun is along the ecliptic, which is inclined $23\frac{1}{2}°$ to the celestial equator along which time is measured.

The amounts due to the first cause are given for 15-day periods in the columns headed "I." When the sun is *ahead* of its average time, the number is +, when behind −.

1. Graph these numbers to a convenient scale on cross-lined paper, and draw through the points graphed a smooth free-hand curve.

Date		I Minutes	II Minutes	Date		I Minutes	II Minutes
Jan.	1	0.00	+3.75	July	1	0.00	+3.75
	16	+2.50	+7.50		16	−2.50	+7.50
Feb.	1	+3.75	+8.50	Aug.	1	−5.00	+9.50
	16	+5.50	+8.50		16	−6.50	+9.50
Mar.	1	+6.50	+7.00	Sept.	1	−7.50	+7.50
	16	+7.50	+1.25		16	−8.50	+2.50
April	1	+8.25	−5.00	Oct.	1	−8.55	−5.00
	16	+8.25	−8.50		16	−8.50	−8.75
May	1	+7.50	−10.00	Nov.	1	−7.50	−10.00
	16	+6.25	−9.00		16	−6.50	−8.75
June	1	+5.00	−7.00	Dec.	1	−5.00	−6.25
	16	+2.25	−2.00		16	−2.50	−2.00

2. The amounts due to the second cause are in the columns headed "II." Graph these values on the same axes and to

the same scale as was used for the data of columns "I," and draw through the points a smooth free-hand curve.

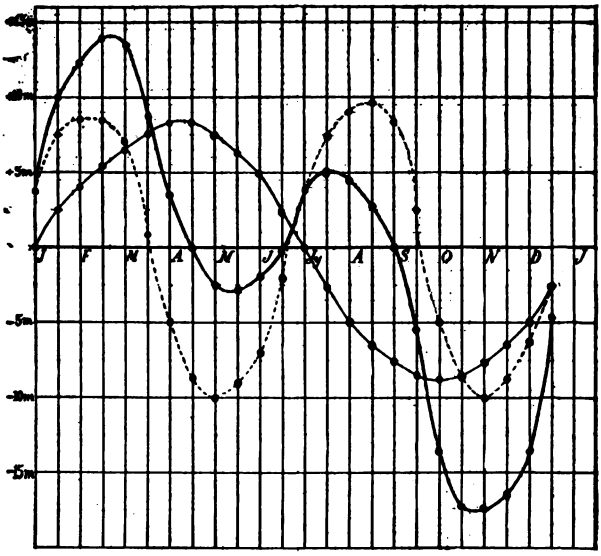


FIG. 203

3. Add, algebraically, the corresponding verticals of the two curves and draw a curve, through the points located by the sums of the verticals, showing the combined effect of the two causes operating together.

The Evaluation of Expressions*

139. The formulas used in the following problems express laws of *motion and mensuration.*

* The general equations of this section might well be pictured by means of diagrams on cross-lined paper or cross-ruled blackboard, or by measuring drawings made on unruled paper. Many of the laws may be graphed with profit.

1. The distance traversed by a moving body is equal to the rate multiplied by the time; that is, $D=rt$.

Find D, if (1) $r=30$ ft. per second, and $t=5$ seconds;
(2) $r=5$ mi. per hour and $t=17$ hours.

2. The area of a rectangle is equal to the product of the base and the altitude; that is, $A=ba$.

Find A, if (1) $b=13$ ft., and $a=24$ feet;
(2) $b=10.2$ in., and $a=3.5$ inches.

3. The area of a triangle is equal to $\frac{1}{2}$ the product of the base by the altitude; that is, $A=\frac{1}{2}ba$.

Find A, if (1) $b=12$ ft., and $a=16$ feet;
(2) $b=8.2$ rd., and $a=7.78$ rods.

4. The area of a parallelogram is equal to the product of the base by the altitude; that is, $A=ba$.

Find A, if (1) $b=28$, and $a=19$;
(2) $b=16.3$, and $a=14.6$.

5. Find the numerical value of each of the following expressions, for $a=5$, $b=3$, $c=10$, $m=4$, $n=1$;

(1) $5cm^2$
(2) $3a^3c$
(3) $13b^2cn^2$
(4) n^5+1
(5) $5abc+3m^2n^3$
(6) $\dfrac{c+2m}{c-2m}$
(7) $a^2+b^2+c^2+m^2+n^2$
(8) $2ab+4bc+5cm^2$
(9) $a^3-b^3+c^3$
(10) $3abcmn$.

6. Two weights, w_1 and w_2, will balance on a beam that lies across a stick when the distances, d_1 and d_2, of the weights from the stick are in the inverse ratio of their weights; i. e., when $\dfrac{d_1}{d_2}=\dfrac{w_2}{w_1}$.

Find d_1, if (1) $d_2=18$ ft., $w_2=60$ lb., $w_1=50$ lb.
(2) $d_2=27$ in., $w_2=36$ lb., $w_1=24$ lb.

Find d_2, if (1) $d_1=40$ in., $w_2=16$ lb., $w_1=18$ lb.
(2) $d_1=25$ in., $w_2=3.8$ lb., $w_1=2.85$ lb.

Find w_1, if (1) $d_1=2.5$ in., $d_2=7.5$ ft., $w_2=10.5$ lb.
(2) $d_1=6.6$ ft., $d_2=9.9$ ft., $w_2=17$ lb.

Find w_2, if (1) $d_1=3.5$ ft., $d_2=8.5$ ft., $w_1=30$ lb.

7. The weight, w, of any mass is equal to the volume, v (number of cubic inches it holds), multiplied by the density (the weight of one cu. in.); i. e., $w=vd$.

Find w, if $v=64$ cu. in., and $d=16.2$ lb.
Find v, if $w=648$ lb., and $d=16.2$ lb.
Find d, if $w=800$, and $v=160$.

8. The weight, w, that a force, p, pulling up a smooth slope h feet high and l feet long will just move, is given by

$$w=\frac{l}{h}p\,.$$

Find h, if $w=120$ lb., $l=20$ rd., $p=60$ lb.

9. A stone, falling from rest, goes in a given time 16 ft. multiplied by the square of the number of seconds it has fallen; i. e., $s=16t^2$.

Find s, if (1) $t=4$ seconds
(2) $t=11.5$ seconds.

Find t, if (1) $s=64$ feet
(2) $s=1{,}600$ feet.

10. A stone, thrown downward, goes in a given time 16 ft. multiplied by the square of the number of seconds it has fallen, plus the product of the velocity with which it is thrown and the number of seconds fallen: i. e., $s=16t^2+vt$.

Find s, if (1) $t=12$ seconds, and $v=3$ ft. per second
(2) $t=8$ seconds, and $v=7$ ft. per second.

Find v, if (1) $t=5$ seconds, and $s=500$ feet.
(2) $t=1.5$ seconds, and $s=60$ feet.

11. The time, t, taken for a pendulum to make a single vibration equals $\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum in feet, g is 32, and π is $\frac{22}{7}$.

Find t, if (1) $l=8$ ft.
(2) $l=\frac{882}{121}$ feet.

Find l, if (1) $t=1$ second
(2) $t=4$ seconds.

12. The area, A, of a triangle is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}\,,$$

where a, b, c represent the lengths of the sides and $s=\frac{1}{2}(a+b+c)$.

Find A, if (1) $a=3,\ b=4,\ c=5$
(2) $a=5,\ b=12,\ c=13$.

13. $h=\dfrac{v^2}{2g}$. Find h, if (1) $v=11,\ g=32.2$
(2) $v=2,\ g=32.2$.

14. $R=\dfrac{rr'}{r+r'}$. Find R, if (1) $r=11.5,\ r'=6.5$
(2) $r=13,\ r'=15$.

15. $E=\dfrac{MV^2}{2}$. Find E, if (1) $M=12,\ V=5$
(2) $M=11,\ V=9$

Find M, if (1) $E=8,\ V=4$
(2) $E=50,\ V=5$.

16. $V=\frac{1}{3}h(B+b+\sqrt{Bb})$.
Find V, if (1) $h=9,\ B=16,\ b=4$
(2) $h=6,\ B=16,\ b=9$.

140. The formulas used here express properties of *the circle and the sphere.*

1. The circumference of a circle is equal to $\frac{22}{7}$ of the diameter; i. e., $C=\pi\cdot d$, where $\pi=\frac{22}{7}$.

Find C, if (1) $d=21$ ft.
(2) $d=7$ ft.
(3) $d=\frac{1}{2}$ ft.

Find d, if (1) $C=88$ ft.
(2) $C=66$ ft.
(3) $C=16$ ft.

2. If r stands for the radius of a circle and C for its circumference, from problem 1, show that $C=2\pi r$.

Find C, if (1) $r=3$ ft. (2) $r=6$ ft. (3) $r=18$ ft.

Find r, if (1) $C=132$ in. (2) $C=66$ ft. (3) $C=110$ rd.

3. The area of a circle equals $\frac{22}{7}$ times the square of the radius; i. e., $A=\pi r^2$, where $\pi=\frac{22}{7}$.

Find A, if (1) $r=3$ ft. (2) $r=7$ ft. (3) $r=21$ ft.

Find r, if (1) $A=154$ sq. ft. (2) $A=220$ sq. ft. (3) $A=86.4$ sq. ft.

4. The volume of a sphere equals $\frac{4}{3}\pi$ times the cube of the radius; i. e., $V=\frac{4}{3}\pi r^3$, where $\pi=\frac{22}{7}$.

Find V, if (1) $r=\frac{7}{2}$ ft. (2) $r=11$ ft. (3) $r=2$ ft.

Find r, if (1) $V=4\frac{4}{21}$ cu. in. (2) $V=33\frac{11}{21}$ cu. ft. (3) $V=51\frac{6}{11}$ cu. rd.

5. The circumference, C, the area, A, of a section through the center, the area, S, of the surface, and the volume, V, of a sphere, are connected with the radius, r, of the sphere by the formulas:

$C=2\pi r$; $A=\pi r^2$; $S=4\pi r^2$; and $V=\frac{4}{3}\pi r^3$.

Find C, A, S, and V for a sphere whose diameter is 1 yard.

Find A, S, V, and r, for $C=18\pi$.

Find C, S, V, and r, for $A=36\pi$.

Find C, A, V, and r, for $S=100\pi$.

Find C, A, S, and r, for $V=36\pi$.

The Right Triangle

141. The sides of a right triangle are connected by a simple law.

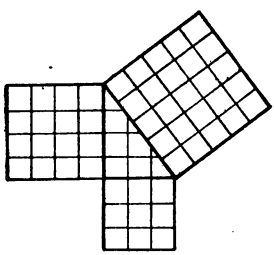


FIG. 204

1. Fig. 204 shows a right triangle whose sides are 3, 4, and 5. How does the sum of the squares on the sides including the right angle compare with the square on the side opposite the right angle (the *hypotenuse*)?

2. How does the sum of the

squares on the sides of the right triangle of Fig. 205 compare with the square on the hypotenuse?

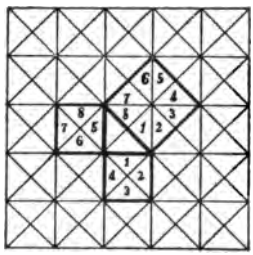


FIG. 205

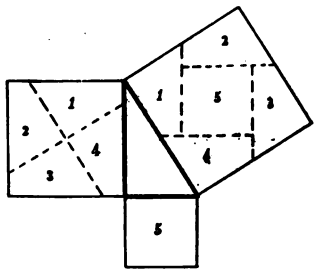


FIG. 206

3. Answer a similar question for the right triangle of Fig. 206.

4. Express this relation for the triangle of Fig. 206 by means of an equation.

142. THEOREM: *In any right triangle the sum of the squares on the sides including the right angle is equal to the square on the hypotenuse.*

1. The two short sides of a right triangle are 6 and 8. What is the length of the hypotenuse?

2. A wall is 16 ft. long and 12 ft. high. How long is a string stretched from a lower corner to the opposite upper corner?

3. The hypotenuse of a right triangle is 30 ft., and one of the other sides is 18 ft. long. How long is the third side?

4. The sides of a right triangle are as 3 is to 4 and the hypotenuse is 35 ft. long. How long are the sides?

5. A ladder 20 ft. long just reaches a window 16 ft. above the ground. How far is the bottom of the ladder from the foot of the wall if the ground is level?

6. The diagonal of a square is 12. What is the side? The perimeter?

The square root of a number, as 80, or a, or $x+y$, that is not a perfect square, is indicated thus: $\sqrt{80}$, or $\sqrt{a}$, or $\sqrt{x+y}$.

7. The diagonal of a square is a. Find the perimeter.

8. Find the hypotenuse, h, of a right triangle whose area is 54 sq. ft. and whose base is 12 feet.

9. Find the hypotenuse, h, of a right triangle whose area is r sq. rd. and whose base is b rods.

10. Find the perimeter of a right triangle whose area is 216 sq. rd. and whose base is 48 rods.

11. Find the perimeter of a right triangle whose area is s sq. rd. and whose base is b rods.

12. A tree, standing on level ground, was broken 45 ft. from the top and the top struck the ground 27 ft. from the stump, the broken end remaining on the stump. How tall was the tree before breaking?

13. Under conditions of problem 12, suppose the top piece t ft. long and that the top struck the ground f ft. from the stump. How tall was the tree before it broke?

The Circle

143. A *circle* is a curve, all of whose points are at the same distance from a fixed point, as C. The length of the curve is the *circumference*. The fixed point C, is the *center* of the circle.

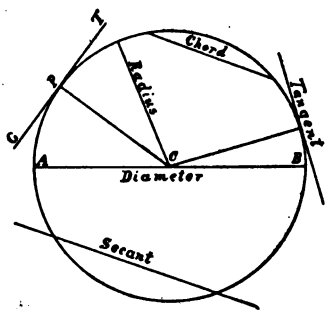


FIG. 207

A *chord* is a line connecting two points of a circle.

The word, *line*, here means *straight line*.

A *radius* is a line connecting the center with a point on the circle.

A *diameter* is a chord through the center.

Prove that a diameter is twice as long as a radius.

A *secant* is a line cutting the circle in two points.

A *tangent* is a line that touches, but does not cut, the circle, however far produced. The point, as P, where a tangent touches the circle is called the *point of contact*, or the *contact-point.*

A radius, drawn to the contact-point of a tangent, is perpendicular to the tangent.

1. Find the angle between the tangent, P T, and the chord P C, if the angles may be designated as shown in Fig. 208.

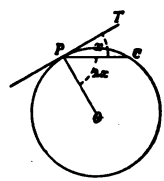


FIG. 208

2. Draw a square, with its sides tangent to a circle, and show that the area of a circle is less than $4r^2$.

A square whose sides are tangent to a circle is a *circumscribed square.*

A square whose sides are chords of a circle is an *inscribed square.*

3. Draw a square inside a circle, with the corners on the curve, and show that the area of a circle is greater than $2r^2$.

The expression, $2r^2 < A < 4r^2$ is read: "Two r-square is less than A is less than four r-square." It means that A is between $2r^2$ and $4r^2$ in magnitude.

The symbol, $<$, means *is less than.*

The symbol, $>$, means *is greater than.*

The length of the curve of a circle is about $3\frac{1}{7}$ times the diameter. More accurately, the ratio is 3.14159. For convenience, the symbol π (called pī) is often used to denote the correct value of the ratio of the circle to its diameter.

The Triangle

144. To find the distance straight through a hill from A to B, a point, C, was chosen on level ground, so that the

distances, A C and B C, could be measured. C D, in the prolongation of B C, was made equal to B C, and C E, in the prolongation of A C, was made equal to A C. The points, D and E, were marked with stakes, and D E was measured and found to be 68.5 rods.

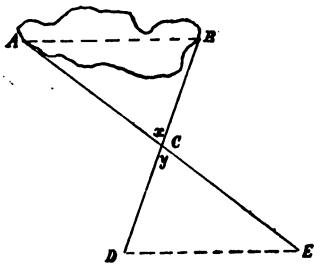


FIG. 209

1. How long is A B?

2. What is the relation of angles x and y?

3. Recalling how D and E were located, what parts (*angles* and *sides*) of triangle A B C are known to be equal to certain parts of triangle C D E?

4. What relation as to form and size do the whole triangles A B C and C D E seem to have?

145. THEOREM: *If two sides and the included angle of one triangle are equal, each to each, to two sides and the included angle of another triangle, the triangles are equal in all respects.*

Draw a triangle, as A B C, and make another triangle as A′B′C′, having $a'=a$, $c'=c$, and angle B=angle B'.

Trace triangle A B C through a piece of thin paper, or on tracing paper, and fit the trace over triangle A′B′C′ by placing the trace of a along a', of B on B′, and of c on c'. What seems to be true of the relations of b and b'? Of $\angle A$ and $\angle A'$? Of $\angle C$ and $\angle C'$?

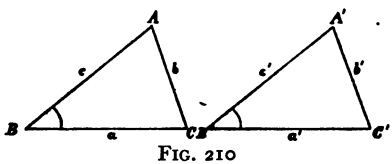


FIG. 210

It is not necessary always actually to do all this fitting and measuring. It is quicker and better *to reason out* the equality of the triangles, thus:

Imagine triangle A B C carried along and placed over tri-

angle A′B′C′, so that point B falls on point B′, and a lies along a'.

As angle B=angle B', c falls along c'.

As $a=a'$, point C must fall on point C′.

As $c=c'$, point A must fall on point A′.

b and b' must then be one and the same straight line (i. e., they must coincide), because *only one straight line can be drawn connecting two points.*

Consequently, the two triangles, A B C and A′B′C′, can be made to fit, or coincide throughout and hence they are equal in all respects, or

$$\begin{array}{lll} \text{If} & a=a', & \text{then,} \quad b=b', \\ & c=c', & \qquad \angle A=\angle A', \\ \text{and} & \angle B=\angle B', & \text{and} \quad \angle C=\angle C'. \end{array}$$

1. If $a=a'$, $b=b'$, and $\angle C=\angle C'$, compare c and c', $\angle A$ and $\angle A'$, $\angle B$ and $\angle B'$.

2. If $b=b'$, $c=c'$, and $\angle A=\angle A'$, compare a and a', $\angle B$ and $\angle B'$, $\angle C$ and $\angle C'$.

3. With the triangles of Fig. 211 prove that

(1) if $a=a'$, $c=c'$, and $y=y'$; then $b=b'$, $x=x'$, and $z=z'$

(2) if $c=c'$, $b=b'$, and $x=x'$; then $a=a'$, $z=z'$, and $y=y'$

(3) if $a=a'$, $b=b'$, and $z=z'$; then $c=c'$, $x=x'$, and $y=y'$.

A triangle that has two equal sides is an **isosceles triangle.**

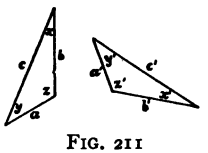


FIG. 211

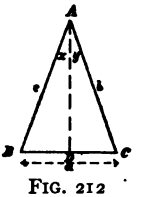


FIG. 212

4. Draw the bisector of the angle A of the isosceles triangle, Fig. 212 ($c=b$), and prove that angle B=angle C.

5. Prove that the angle-bisector of angle A bisects a, at D.

6. Prove that the angle-bisector of A is perpendicular to side a.

7. Prove that if the sides including the right angle of a right triangle are equal, each to each, to the sides including the right angle of another right triangle, the triangles are equal in all respects.

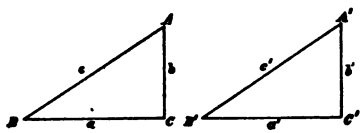


FIG. 213

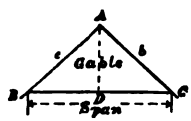


FIG. 214

8. The span of a roof is 48 feet, and c and b are 32-foot rafters. What is the rise A D? (A D is the angle-bisector of A.)

9. The span of a roof is $2s$, the length of the rafters, r. Find the rise h.

146. A triangle that has all sides equal is an **equilateral triangle.**

1. Prove that an equilateral triangle is equiangular (all angles equal). Apply problem 4, p. 203.

2. The length of a side of an equilateral triangle is a. Find the length of the angle-bisectors, shown in Fig. 215.

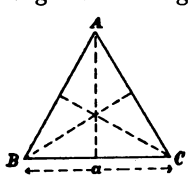


FIG. 215

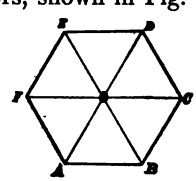


FIG. 216

3. How many degrees are there in one of the angles of an equilateral triangle?

Let x denote the value of one of the angles, then $3x = 180°$. Why?

4. How many tiles of the shape of equilateral triangles will just cover the plane if placed as shown in Fig. 216 around a point as O?

5. Draw 6 equal equilateral triangles in positions as shown in Fig. 217. What kind of figure is formed?

Compare the lengths of the sides of the large figure.

Compare the angles of the large figure.

Compare the distances from O to each of the corners A, B, C, etc.

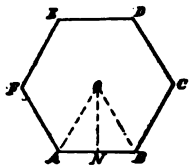


FIG. 217

A plane figure that is bounded by 6 equal sides making equal angles is a **regular hexagon.**

6. If O were used as a center and a circle were drawn with O A as a radius, through what points would the circle pass?

7. How many chords equal in length to the radius of a circle, if placed end to end, will reach around the circle?

8. How may a regular hexagon be readily constructed by the aid of a circle?

9. Six lights are placed equal distances apart around a circle of radius r. Find the distances from any one light to each of the others.

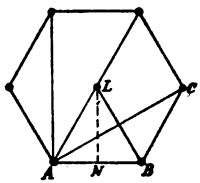


FIG. 218

L B and A C bisect each other perpendicularly. The distance L A, Fig. 218, is the radius of the regular hexagon.

The distance L N perpendicular to A B, Fig. 218, is the **apothem.**

10. Find the area of an equilateral triangle whose side is 12.

11. Find the area of an equilateral triangle whose side is a. Fig. 219.

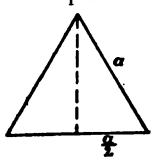


FIG. 219

12. Find the area of a regular hexagon, one of whose sides is 10.

13. Find the area of a regular hexagon (Fig. 220), one of whose sides is s.

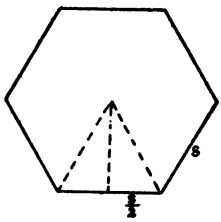


FIG. 220

14. A wheel of 85 cogs works into a wheel of 27 cogs. In how many revolutions, x, of the larger does the smaller gain g revolutions?

15. A wheel of a cogs works into a wheel of b $(a>b)$ cogs. In how many revolutions, x, of the larger, does the smaller gain g revolutions?

16. Draw two concentric circles of radii 6 and 10. Find the length, l, of the chord of the outer circle that is tangent to the inner circle.

A radius, drawn to the contact-point of a tangent, is perpendicular to the tangent. See p. 201.

17. Draw two concentric circles of radii r and R $(r<R)$. Find the length, l, of the chord of the outer circle, that is tangent to the inner circle.

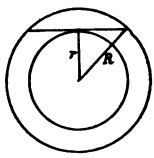


FIG. 221

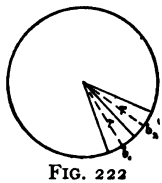


FIG. 222

If the area of a circle be divided by drawing radii, as in Fig. 222, into small triangles, having curved bases, the sum of the areas of all these triangles equals the area of the circle.

18. Denoting the radius of the circle by r, and the small bases by b_1, b_2, b_3, b_4, etc., express the sum, S, of the areas of the triangles.

19. To what curved line is the sum $b_1+b_2+b_3+\ldots\ldots$ equal?

The series of small dots, ... , means "and so on." It is called a *symbol of continuation.*

20. Show that the area, A, of the circle is given by

$$A=S=\frac{r}{2}\cdot(b_1+b_2+b_3+\ldots\ldots)=\frac{r}{2}\cdot C=\frac{r\cdot C}{2}=r\cdot\frac{C}{2},$$

C denoting the length of the circle.

21. State in words the meaning of

1. $A=S$
2. $A=r\cdot\frac{C}{2}$
3. $A=\frac{r}{2}\cdot C$
4. $A=\frac{r\cdot C}{2}$.

Summary

Algebraic expressions are only translations of verbal statements of number relations into compact symbolic forms.

Algebraic expressions need often to be translated into verbal forms and verbal expressions of number relations into algebraic or symbolic forms.

The fundamental operations of arithmetic may be symbolized thus:

$s=a_1+a_2+a_3\ldots+a_n$	for addition of n addends
$d=m-s$	for subtraction
$P=\mathrm{M}\cdot m$	for multiplication
$Q=D\div d$	for division.

In division, when there is no remainder, $D=Qd$.

In division, when there is a remainder, $D=Qd+r$.

Adding two fractions is indicated by $\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_1d_2+n_2d_1}{d_1\cdot d_2}$.

Subtracting fractions is shown by $\frac{n_1}{d_1}-\frac{n_2}{d_2}=\frac{n_1d_2-n_2d_1}{d_1\cdot d_2}$.

Multiplying two fractions is shown by $\frac{n_1}{d_1}\times\frac{n_2}{d_2}=\frac{n_1n_2}{d_1d_2}$.

Dividing one fraction by another is shown by $\frac{n_1}{d_1}\div\frac{n_2}{d_2}=\frac{n_1}{d_1}\times\frac{d_2}{n_2}$.

Percentage is shown by $p=\frac{br}{100}$.

Interest is shown by $i=\frac{prt}{100}$.

Percentage and *interest problems* may be solved graphically.

Statistical laws may be shown graphically.

Problems in *motion* and *mensuration* may be solved by substituting in the formulas that express the laws of motion and mensuration.

Problems on *the circle and the sphere* may be solved by substituting in the formulas that express the properties of the circle and the sphere.

In a *right triangle* the square of the hypotenuse is equal to the sum of the squares on the other sides.

A *circle* is a plane curve all of whose points are at the same distance from a fixed point, called the *center*.

The *circumference* is the length of the circle.

A *chord* is a straight line connecting two points of the circle.

A **radius** (plural rā′dĭ-ī) of a circle is a straight line connecting the center with a point on the circle.

A **diameter** is a chord through the center.

A **secant** is a straight line cutting the circle in two points.

A **tangent** is a line that touches but does not cut the circle, however far produced.

A **circumscribed square** is a square whose sides are tangent to a circle.

An **inscribed square** is a square whose sides are chords of a circle. The symbol $<$ means *is less than*, and $>$ means *is greater than*.

If two sides and the included angle of one triangle are equal, each to each, to two sides and the included angle of another triangle, the triangles are equal in all respects.

A **regular hexagon** is a plane figure that is bounded by 6 equal sides, making equal angles.

The **apothem** of a regular hexagon is the perpendicular distance from the center to the side of the hexagon.

The side of a regular hexagon is equal to the radius of the circumscribed circle.

The **radius** of the **hexagon** is the *radius* of the *circumscribed circle.*

The **symbol of continuation** is . . . . . , meaning, *and so on according to the same law.*

CHAPTER X

THE SIMPLE EQUATION IN ONE UNKNOWN

147. Algebraic expressions may be pictured or graphed, and equations may be regarded as having been obtained by giving to the expressions the particular value 0.

1. A man walks along a straight road, A B, in the direction of the arrows (→) at the rate of 3 mi. an hour. How far from the house, H, is he—

4 hr. before reaching it?	1 hr. after reaching it?
3 hr. before reaching it?	2 hr. after reaching it?
2 hr. before reaching it?	3 hr. after reaching it?
1 hr. before reaching it?	4 hr. after reaching it?

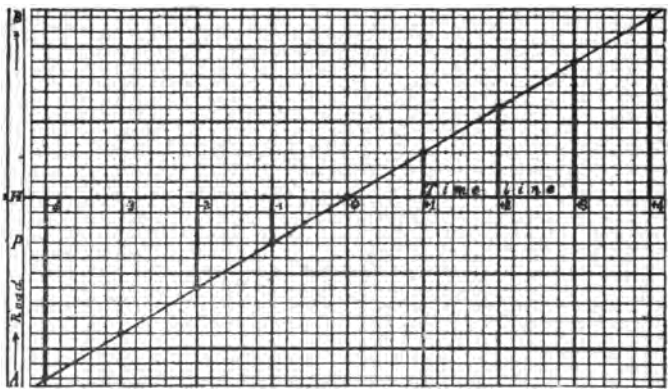


FIG. 223

2. Show the times to a scale (1 large space = 1 hour) on a horizontal line, and at each time draw, or measure, a vertical downward to represent to scale (1 small space = 1 mile) his distances below H, and draw verticals upward to represent his distances above H.

The row of points at the ends of the verticals pictures the successive positions of the man with respect to the house, H.

3. Show on the graph how far the man was from the house at $t=+\frac{1}{2}$ hr. At $t=-\frac{1}{2}$ hr. At $t=0$.

4. Show on the graph when the man was 7 miles below H; 4 miles above H.

148. By putting in dots more thickly more distances from H would be represented and the line through all the points represents all distances of the man from H between -4 hr. and $+4$ hr.

Since the distances from H for all the times from t hours before to t hours after arriving at H are also represented by

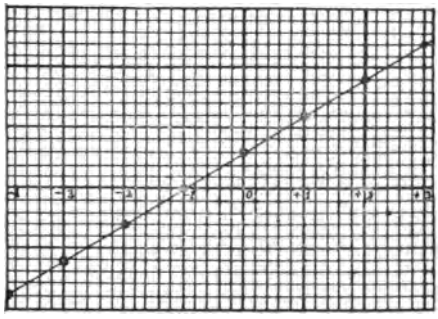


FIG. 224

$3t$, the line drawn through the row of points also pictures the expression $3t$. The line through the points is called the **graph** of $3t$.

If the distances of the traveler were expressed from a point, P, 3 miles below H, the row of points and the line through them would be as in Fig. 224. The algebraic expression of these same distances is $3t+3$. The line through the dots of Fig. 224 is the **graph** of $3t+3$.

149. Of course, any other letter, as x, might be used to stand for the times. If x had been used instead of t, the line of dots of Fig. 224 would have been the graph of $3x+3$.

The distances that are represented by the verticals might have been found by substituting -4, -3, -2, -1, 0, $+1$, $+2$, $+3$, $+4$ in turn for x, calculating the corresponding values -9, -6, -3, 0, $+3$, $+6$, $+9$, $+12$, $+15$, of $3x+3$, and then representing these numbers to scale on the verticals. *Graphs* may be drawn for any expressions in the same way.

1. Draw a graph of $3t-3$; of $2t+3$; of $2t-3$; of $3t+2$; of $2t-2$.

2. Draw graphs of the following expressions:

(1) $2x+1$	(4) $3x-4$	(7) $2x+4$
(2) $2x-1$	(5) $4x-3$	(8) $4x+2$
(3) $3x+4$	(6) $4x+3$	(9) $2x-4$.

150. The graphs show:

(1) that there are a great many values of any one of these expressions, one, indeed, for every value of t, or of x, we care to substitute;

(2) that there is but *one* place, or point, where the value of the expression, as $3x+3$, is 0;

(3) that the value of x that makes $3x+3$, or other like expression, 0, is the distance from O in the graph to the crossing-point of the line of points and the horizontal line.

This same value of x that makes $3x+3$ equal to 0 can be more quickly found by solving the equation, $3x+3=0$, for x, thus:

$$3x+3=0$$
$$3x=-3$$
$$x=-1$$

CHECK: $3\cdot(-1)+3=-3+3=0$.

151. When any equation as $3x-6=0$ is written down, it may be regarded as merely symbolizing the question:

What value, or what values, if any, of x are there, which substituted in $3x-6$ will make $3x-6$ equal to 0?

Two kinds of equation are used in algebra:

I. The **identical equation,** or the **identity,** is an equation that is true for any, or all, values of the letters.

EXAMPLES: 1. $(x+3)(x-3)=x^2-9$.
2. $(x+a)(x+a)=x^2+2ax+a^2$.

Test whether example 1 is true for $x=4$; for $x=7$; for $x=10$; for $x=1$.

Test whether example 2 is true for $x=3$, $a=1$; for $x=5$, $a=-2$; for $x=12$, $a=10$.

Identical equations are sometimes written with the sign $\equiv$, when it is desired to distinguish them from other kinds of equations.

II. The **conditional equation** is an equation that is true only for one or for a definite number of values of its letters.

EXAMPLES: 1. $5x-15=25$.
2. $x^2-16=0$.
3. $(x-1)(x+2)(x-3)=0$.

Test whether example 1 is correct for $x=8$; for any other value of x.

Test whether example 2 is correct for $x=4$; for $x=-4$; for any other values of x.

Test whether example 3 is correct for $x=1$; for $x=-2$; for $x=3$; for any other values of x.

When nothing is said to the contrary, the word "equation" means **conditional equation.**

The *left side* of an equation is called the **left member,** or the **first member,** and the *right side,* the **right member,** or the **second member.**

Point out which of the following equations are (1) identical equations, or identities, (2) conditional equations, and give reasons for your answers:

1. $x-1=0$
2. $x^2-4=0$
3. $(x-2)^2=x^2-4x+4$
4. $x^2-16=(x+4)(x-4)$
5. $a(x-3)=ax-3a$
6. $a(x-3)=a$
7. $x(x-1)=x^2-x$
8. $x(x-1)=0$
9. $x^3-27=0$
10. $a(b-5)=ab-5a$
11. $(a+x)(a-x)=a^2-x^2$
12. $(a-3)(a+4)=0$
13. $(a-3)(a+4)=a^2+a-12$
14. $\frac{c+8}{4}=\frac{c}{4}+2$
15. $\frac{c-7}{5}=0$
16. $m^4-n^4=(m^2+n^2)(m^2-n^2)$.

While any letter may be used to denote the *primary unknown* (or primary variable), it is customary to use x, or y, or z, or some one of the later letters of the alphabet for this purpose.

The earlier letters of the alphabet, as a, b, m, l, k, etc., are quite commonly used to denote *fixed*, or *constant*, numbers, like 6, 15, 75, etc. The letters that abbreviate words, as i for interest, b for base, are also much used to denote numbers.

Stating Algebraic Problems

152. The problems of the foregoing chapters of this book, as well as those of modern arithmetic, show that letters may be used to advantage to express numbers, and to simplify stating and solving problems. Algebraic skill means nearly the same thing as skill in stating and solving problems. This skill can be acquired only through much practice, and not by committing rules and directions. Still, it may aid to know that in algebra *stating a problem* usually means expressing in the language of algebraic symbols certain number-relations that are given in verbal language.

The **statement of a problem** in most cases takes the form of an equation. To obtain this equation, the following may serve to guide the beginner:

I. *Denote the unknown number by a letter, then translate the verbal statement of the number-relations into a symbolic statement in equation form.*

The first letters of words are convenient letters to use to denote numbers, while they are yet unknown; as n for number, t for time, a for age, etc.

Give statements of the following problems, then solve and check them.

1. Two-thirds of a number diminished by 12 equals 4. Find the number.

STATEMENT:

Two-thirds of a number diminished by 12 equals 4.

$$\tfrac{2}{3} \times n - 12 = 4.$$

Solve the equation and check by substitution.

2. Twice a number decreased by 12 is the same as the number increased by $\frac{3}{4}$ of itself. Find the number.

3. Six times a number increased by $\frac{1}{2}$ of itself equals 11. Find the number.

4. Four times a number increased by $\frac{1}{5}$ of itself is the same as twice the number increased by 11. Find the number.

5. The acute angles of a right triangle are $4x$ and $5x$. Find the acute angles.

6. One of two supplementary adjacent angles is 3 times the other. Find the two angles.

7. The angles for a triangle are x, $2x$, and $3x$ degrees. Find the numerical values.

Another guide that is often useful in stating a verbal problem in equation form is the following:

II. *By the aid of literal and of arithmetical numbers express some number of the problem in two different forms, and write the two expressions equal to one another.*

Thus an equation is a statement, in symbols, that two different number-expressions stand for the same number.

8. A rectangle is 3 times as long as wide, and the perimeter is 48 inches. Find the width and length.

Equate two different expressions of the perimeter.

9. Double the number of years in a boy's age is 16 more than his age 2 years ago. How old is the boy?

Equate two different expressions for the difference of double the boy's present age and his age two years ago.

10. The difference of double a boy's age and 3 times his age 10 years ago is his present age. How old is the boy?

STATEMENT IN SYMBOLS:

	$2x-3(x-10) = x$	(1)
$-3(x-10) \equiv -3x+30$,	hence $2x-3x+30 = x$	(2)
$2x-3x \equiv -x$,	hence $-x+30 = x$	(3)
Adding x to both sides	$30 = 2x$	(4)
Dividing by 2	$15 = x$.	(5)

Check by testing in the conditions of the problem.

The boy's age is 15 years.

153. The solution of an equation consists in getting another equation in which the unknown stands alone in one member, with the numbers in terms of which the unknown is to be expressed, in the other member. The required changes are made by means of the axioms stated on p. 26, §**21**, or by means of the fifth axiom now added below. The five axioms that are to be the reasons for changes in equations, are now stated in final form.

I. *If the same number, or equal numbers, be added to equals the sums are equal.* (*Addition-axiom.*)

II. *If the same number, or equal numbers, be subtracted from equal numbers, the remainders are equal.* (*Subtraction-axiom.*)

III. *If equal numbers be multiplied by the same number, or by equal numbers, the products are equal.* (*Multiplication-axiom.*)

IV. *If equal numbers be divided by the same number, or by equal numbers, the quotients are equal.* (*Division-axiom.*)

Division by zero, or by any expression in which the letters have values that make the divisor zero, is not permitted.

V. *Any change may be made in the form of an expression, that does not change its value.*

One of the most important form-changes is the removal and introduction of parentheses.

When a reason for a change is asked, the pupil should be able to quote, or to cite, the axiom that justifies the change.

154. Give the reason for the truth of the conclusion in each of the following:

1. $a=9$ and $b=3$, then $a+b=12$
2. $m=13$ and $n=2$, then $mn=26$
3. $a=x$ and $b=y$, then $a+b=x+y$
4. $c=16$ and $d=4$, then $c-d=12$
5. $m=28$ and $n=7$, then $\frac{m}{n}=4$
6. $\frac{c}{12}=5$ and $\frac{d}{12}=3$, then $\frac{c+d}{12}=8$.

Give reasons for the following:

7. If $a=7$, then $5a=35$
8. If $\frac{f}{16}=3$, then $f=48$
9. If $mn=6n$, then $m=6$
10. If $2y=18$, then $y=9$
11. If $x-5=3$, then $x=8$
12. If $\frac{r}{10}=7$, then $r=70$
13. If $ax=b$, then $x=\frac{b}{a}$.

155. *To solve an equation means to find the number, or the numbers, which, substituted for the unknown in the equation, reduces both members to the same number.*

Any number that fulfills these conditions is a **root of the equation.**

The solution of an equation means two different things in mathematics, viz.:

(1) The solving of the equation.

(2) The value, or values, found by solving the equation.

In this book the word "solution," from now on, usually has the second meaning.

To check the process of solving an equation means to show that the work of *finding the unknown is correct.*

This is often done by checking the "solution," i. e., the *result*, though it is generally desirable to examine the reasoning, or to use an independent process.

Distinguish between *root of a number* (pp. 24 and 28) and *root of an equation.*

To check the *solution of an equation* means to show that the result is a root of the equation. This is commonly done by substituting.

To check the *solution of a problem* means to show that the result answers the conditions stated in the problem. To do this the result must be tested by the stated conditions, and not merely by substituting in the equation solved. This equation may itself be wrong.

Exercises for Practice

156. It should be borne in mind that finding the *root* of an equation includes checking the *result.*

Find the roots of the following equations:

1. $5x+9=3x+17$.

FULL SOLUTION:

	$5x+9=3x+17$	
Subtract $3x$	$3x \quad =3x$	(by Axiom II)
	$2x+9= \quad 17$	
Subtract 9	$9= \quad 9$	(Ax. II)
	$2x \quad = \quad 8$	
Divide by 2	$x \quad = \quad 4$	(Ax. IV)

CHECK: $5x+\ 9=5\cdot 4+\ 9=29$

$3x+17=3\cdot 4+17=29$.

Hence, 4 is a root of $5x+9=3x+17$.

SHORTENED SOLUTION:

$$5x+9=3x+17$$

Subtract $3x+9$ from both sides (Ax. II) getting

$$2x=8$$

$$x=4.$$

CHECK: $5\cdot 4+9=29=3\cdot 4+17$.

Hence, 4 is a root of $5x+9=3x+17$.

NOTE: The reasons need not be written, but they should be thought.

In practice only this is needed:

$$5x+9=3x+17$$
$$2x=8$$
$$x=4$$

CHECK: $5\cdot 4+9=29=3\cdot 4+17$.

2. $5x+17=8x-2$
3. $3x+5=5x-15$
4. $6x-7=10x+1$
5. $2x-3=3x-7$
6. $3x+15=x+25$
7. $9x-8=25-2x$
8. $16+3x=6+x$
9. $20-4x=8-10x$
10. $18-x=4x+3$
11. $5x-17+3x-5=x-1$.
    By Axiom V, $8x-22=x-1$
12. $6x-7-8x+115=0$
13. $8x-22=-2x+108$
14. $7x-9-10x=3x-15$
15. $\frac{x}{4}+2x-8=3x-5$
16. $8(x-2)=7x-9$
17. $11-3(x-2)=x-8$
18. $\frac{3x}{4}+5=x+3$
19. $3x-2(x+5)=6x-20$
20. $3(x-2)+15=5x-3$
21. $\frac{3x}{4}+5=91-10x$
22. $5(x-1)=3(x+1)$
23. $8(3-2x)-2(5-x)=28$
24. $-3x-24=33(2-x)$
25. $5(1-13x)=35-105x$
26. $8+2x+\frac{x}{4}=1\frac{3}{4}+\frac{2x}{3}$
27. $3x+14-5(x-3)=4(x+3)$
28. $2(x-1)+3(x-2)=4(x+3)$
29. $2(x+1)+3(x+2)+4(x+3)=110$
30. $5(x-2)+6(x-1)-4(x-5)=60$
31. $\frac{2x+3}{5}-\frac{x-2}{3}=\frac{7}{5}$

32. $\frac{3x-2}{7}-\frac{1-4x}{3}=8\frac{4}{21}$

33. $x(x-1)-x(x-2)=4(x-3)$

34. $\frac{3(5-x)}{4}+\frac{6-x}{5}=2x$

35. $4(2x+9)+3(x-9)=\frac{13(x-5)}{7}$.

Problems Leading to Equations in One Unknown

157. State and solve the following problems and interpret the results:

1. A rectangle is 2 units longer than three times the width, and the perimeter is 60. Find the length and the width.

2. The area of a triangle, whose altitude is 5 units less than the base, is equal to 10 square units less than half the area of a square on the base of the triangle. Find the base and the altitude of the triangle.

3. One-fourth of the difference of 3 times a number and 8 is 10. Find the number.

4. A is 160 yd. east and B 112 yd. west of a gate. Both start at the same time to walk toward the gate, A going 3 yd. and B 2 yd. a second. When will they be at equal distances from the gate?

5. Baking powder is made up of as much starch as soda and twice as much cream of tartar as soda. How much of each ingredient is there in 2 lb. of baking powder?

6. An express train whose rate is 40 mi. per hour, starts 1 hr. 4 min. after a freight train, and overtakes it in 1 hr. 36 min. How many miles per hour does the freight train run?

Let some letter, as x, be the rate of the freight train and notice that the trains go the same distance.

7. Two trains start at the same time from S, one going east at the rate of 35 miles per hour, and the other going west

at a rate $\frac{1}{7}$ faster. How long after starting will they be exactly 100 miles apart?

8. Sound travels 1,080 ft. per second. If the sound of a stroke of lightning is heard $3\frac{1}{2}$ seconds after the flash, how far away is the stroke?

9. If sound travels f ft. per second, how far away is a lightning stroke if the sound is heard s seconds after the flash?

10. A tree half a mile distant was struck by lightning. It took $2\frac{1}{8}$ seconds for the sound to reach the ear. Find the rate in feet per second, at which the sound traveled.

11. A tree m feet distant was struck by lightning. It took t seconds for the sound to arrive. What was the rate in feet per second at which the sound traveled?

12. A man walks beside a railway at the rate of 4 mi. per hour; a train 208 yd. long, running 30 mi. per hour, overtakes him. How long will it take the train to pass the man?

13. A man walks beside a railway at the rate of m mi. per hour. If a train l yd. long and traveling n mi. per hour overtakes him, how long will it take the train to pass the man?

14. Solve the following exercises, as many as possible mentally:

(1) $\frac{l}{5}+6=8$

(2) $\frac{2r}{3}-3=1$

(3) $10s-8=3-s$

(4) $3t+13-5t=t$

(5) $\frac{3x-5}{2}-3=-1$

(6) $10-x=-20$

(7) $\frac{3x-2}{5}+6=2x$

(8) $3x-\frac{1-2x}{5}=-7$

(9) $12-\frac{4x-3}{7}=x+3$

(10) $ax-bx=6a-6b$

(11) $3ax+x=12a+4$

(12) $8-(5-2x)=-x$

(13) $ax-\frac{x-a}{c}=a^2$

(14) $ax-\frac{a(x-c)}{3}=\frac{5ac}{3}$.

15. A train moves at a uniform rate. If the rate were 6 mi. per hour faster the distance it would go in 8 hr. is 50 mi. greater than the distance it would go in 11 hr. at a rate 7 mi. per hour less than the actual rate. Find the actual rate of the train.

16. Two trains go from P and Q on different routes, one of which is 15 mi. longer than the other. The train on the shorter route takes 6 hr., and the train on the longer, running 10 mi. less per hour, takes $2\frac{1}{2}$ hr. Find the length of each route.

17. The distance from A to B is 100 mi. A train leaving A at a certain rate, meets with an accident 20 mi. from B, reducing the speed one-half and causing it to reach B 1 hr. late. What was the rate per hour before the accident?

18. An express train whose rate is r miles per hour, starts h hours after a freight train and overtakes it in t hours. Find the rate per hour of the freight train.

19. Two trains start from the same place at the same time, one going east at the rate of m miles per hour, the other going west at the rate of n miles per hour. How long after starting will they be c miles apart?

20. A man rows downstream at the rate of 6 miles per hour and returns at the rate of 3 miles per hour. How far downstream can he go and return in 9 hours?

21. At what time between 3 and 4 o'clock are the hands of the clock together?

Let x denote the number of minute-spaces over which the minute hand passes from 3 o'clock until it first overtakes the hour hand, then show that $\frac{x}{12}+15=x$, whence, $x=16\frac{4}{11}$.

Hence, the hands are together at $16\frac{4}{11}$ minutes past 3 o'clock.

22. At what time between 2 and 3 o'clock are the hands of the clock together?

23. At what time between 3 and 4 o'clock are the hands of the clock at right angles (two results)?

24. At what time between 7 and 8 o'clock are the hands of the clock pointing in opposite directions?

25. Solve the following for x:

(1) $\frac{x}{a}+3=\frac{1}{2a}$

(2) $\frac{x}{a}+\frac{x}{b}=\frac{1}{ab}$

(3) $-3(x+a)=2(a-3x)$

(4) $a^2x-b^2x=a-b$

(5) $b^2x-a^2x=a-b$

(6) $\frac{x-3}{15}+\frac{x+2}{4}=6$

(7) $x-\frac{-x}{5}-\frac{2}{7}=7\frac{5}{7}$

(8) $\frac{x}{3}+\frac{x}{4}=\frac{x}{6}+4$

(9) $\frac{15x}{2}-\frac{x}{4}=2\frac{1}{3}$

(10) $\frac{x}{c}-x=-c+\frac{1}{c}$

(11) $5x-\frac{3-2x}{2}=2x+2\frac{1}{2}$

(12) $x-\frac{6-x}{2}=2-(x-5)$

(13) $cx-\frac{a^2(x-3)}{c}=\frac{3a^2}{c}-\frac{a-c}{c}$.

26. Venus makes its orbit in 224.7 days or in about $7\frac{1}{2}$ months; the earth starting as shown in the Fig. 225. In how many days will Venus next be in line between the earth and the sun?

The rate per month of Venus is $\frac{2}{15}$ of the orbit, that of the earth $\frac{1}{12}$. Let x be the required number of months.

Express by an equation that Venus must make one more revolution than the earth.

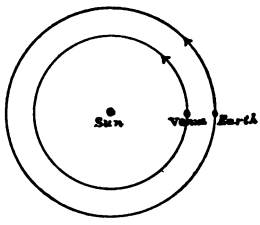


FIG. 225

27. Calling the revolution of Venus about the sun $7\frac{1}{2}$ months, and that of Mercury 3 months, how many months after Mercury is in the line between Venus and the sun will it next be in the same relative position?

28. Seen from the earth, the moon completes the circuit of the heavens in about 27 days, 8 hours, and the sun in 365

days, 6 hours, in the same direction. Required the time, to .001 day, from one full moon to the next, the motion supposed uniform.

29. If a revolution of one planet is a days and of another b days ($a<b$), how many days are there between successive similar relative positions?

30. Use the result of 29 as a formula, and solve problems 25 to 27 by means of it.

31. Using the formula, $t=\pi\sqrt{\frac{l}{g}}$, find how many times as fast as a 2-foot pendulum a 6-inch pendulum vibrates.

32. The difference of the squares of two consecutive numbers is 19. Find the numbers.

33. The difference of the squares of two consecutive numbers is 273. Find the numbers.

34. The difference of the squares of two consecutive numbers is a. Find the numbers.

35. Solve problems 32 and 33, using the result of 34 as a formula.

36. The difference of the squares of two consecutive even numbers is 28. Find the numbers.

37. The difference of the squares of two consecutive even numbers is 100. Find the numbers.

38. The difference of the squares of two consecutive even numbers is a. Find the numbers.

39. Solve problems 36 and 37 using the result of 38 as a formula.

40. The difference of the squares of two consecutive odd numbers is 48. Find the numbers.

41. The difference of the squares of two consecutive odd numbers is s. Find the numbers.

42. Using the result of problem 41 as a formula, find two consecutive odd numbers the difference of whose squares is 56.

43. Solve the following equations (1) for x; and when possible, (2) for a; (3) for z:

(1) $x+a=2a-z$

(2) $3x-2a=z-a$

(3) $\frac{a}{x}=\frac{z}{3x}-a$

(4) $\frac{1}{ax}-\frac{5}{xz}=2$

(5) $axz=12$

(6) $axz=6x-1$

(7) $\frac{a}{x}+\frac{z}{x}=4$

(8) $15x-\frac{6x+1}{2}-\frac{x-1}{3}=-6$

(9) $\frac{a}{6(x-1)}+\frac{1}{x-1}=z-\frac{1}{x-1}$

(10) $\frac{x}{3}-\frac{6(x-1)}{5}-\frac{1-2x}{3}=2\frac{2}{3}$.

44. Bell-metal is by weight 5 parts tin and 16 parts copper. How many pounds of tin and copper are there in a bell weighing 4,800 pounds?

FIRST METHOD: Let x be the number of pounds of tin, then show that $x+\frac{16x}{5}=4{,}800$, and solve for x, and find $\frac{16x}{5}$.

SECOND METHOD (without fractions): Let $5x$ be the number of pounds of tin, and solve: $5x+16x=4{,}800$.

After finding x, calculate $5x$ and $16x$.

45. Gunpowder contains, by weight, 6 parts saltpeter, 1 part sulphur, and 1 part charcoal. How many pounds of saltpeter, of sulphur, and of charcoal are there in 120 pounds of gunpowder?

46. If gunpowder were composed of 4 parts, by weight, of saltpeter, 2 parts sulphur, and 3 parts charcoal, how many pounds of each would there be in 200 pounds of gunpowder?

47. With ingredients as in problem 45, how much saltpeter is burned in the discharge of a cannon using 50 pounds of powder to the cartridge?

48. Baking powder is composed of 4 parts, by weight, of cream of tartar, 1 part starch, and 1 part soda. How much of each ingredient is there in 18 lb. of baking powder?

49. A certain mixture weighing m pounds contains a parts by weight of copper, b parts of iron, and c parts of carbon. How many pounds of each ingredient are there in the mixture?

50. Solve 45, 46, 47, and 48, using the result of 49 as a formula.

51. In a watch-case weighing 2 ounces, the gold is 14 carats fine, i. e., there are 14 parts of gold in every 24 parts of the whole alloy. How many ounces of pure gold are there in the case?

52. In an alloy of silver and copper weighing 90 ounces, there are 6 ounces of copper. Find how much silver must be added so that 10 ounces of the new alloy shall contain $\frac{2}{5}$ ounce of copper.

Let x be the number of lb. of silver added. The new compound then weighs $(90+x)$ lb.

53. A certain compound contains, by weight, 5 parts carbon to every 3 parts of iron, and 7 parts of iron to every 2 parts of copper. In 124 pounds of the compound, how many pounds are there of carbon, of iron, and of copper?

54. If 80 pounds of sea-water contains 4 pounds of salt, how much fresh water must be added to make a new solution of which 45 pounds contain $\frac{2}{3}$ pound of salt?

55. In a mass of alloy for watch-cases, which contains 60 oz., there are 20 oz. of gold. How much copper must be added so that in a case weighing 2 oz. there will be $\frac{1}{2}$ oz. of gold?

56. In an alloy weighing 80 grams, there are 34 grams of gold. How much nickel must be added so that a ring made from the new alloy and weighing $1\frac{1}{2}$ grams shall contain $\frac{1}{4}$ gram of gold?

57. In an alloy weighing a oz. there are b oz. of gold. How much of another metal must be added so that a portion weighing c oz. shall contain d oz. of gold?

58. Solve the following for s:

(1) $s(s-1)-s(s-2)=3$

(2) $(s+1)(s+3)=(s+2)(s+5)-13$

(3) $4(s+6)-2(s-3)=38$

(4) $(s+3)=s+3(s+8)$

(5) $\frac{s-1}{s+1}=\frac{s+7}{s+17}$

(6) $\frac{7s+5}{11}-\frac{s+5}{3}=0$

(7) $\frac{1}{(s+3)(s+6)}=\frac{1}{(s+2)(s+8)}$

(8) $\frac{1}{s-1}-\frac{3}{s-3}=\frac{2}{s-2}-\frac{4}{s-4}$

(9) $\frac{s-5}{s-3}-\frac{s-6}{s-4}=\frac{s+3}{s+1}-\frac{s+4}{s+2}$

(10) $\frac{s-4}{3}+\frac{s+4}{7}=\frac{s+20}{10}+\frac{s-5}{5}$

(11) $-\frac{s-1}{3}-\frac{s-2}{4}=2$

(12) $\frac{2s-3}{4}-\frac{s+5}{2}-\frac{s-1}{2}=-1\frac{3}{4}$.

59. A pound of lead loses $\frac{5}{57}$ of a pound, and a pound of iron loses $\frac{2}{15}$ of a pound when weighed in water. How many pounds of lead and of iron are there in a mass of lead and iron that weighs 159 lb. in air and 143 lb. in water?

If an object weighing 2 lb. in the air is suspended by a cord and weighed when immersed in water, it will weigh less than 2 lb. It can be shown that the loss of weight is the same as the weight of the water the object displaces.

60. A mass of gold weighs 97 oz. in air and 92 oz. in water, and a mass of silver weighs 21 oz. in air and 19 oz. in water. How many ounces of gold and of silver are there in

a mass of gold and silver, that weighs 320 oz. in air and 298 oz. in water?

61. Into what two sums can \$1,000 be divided so that the income of the one at 6% shall equal the income of the other at 4%?

62. How can a man divide \$2,000 so that the income of part at 4% shall be the same as that of the rest at 5%?

63. How many dollars must be invested at 4% to give the same income as that of \$2,500 at 6%?

64. What per cent. of evaporation must take place from a 6% solution of salt and water (salt-water of which 6% by weight is salt) to make the remaining portion of the mixture an 8% solution? A 13% solution?

Let x be the number of per cent. evaporated, then $100-x$ is the per cent. remaining. Then show $.08(100-x)=.06\times 100$.

65. What per cent. of evaporation must take place from a 90% solution to produce a 95% solution?

66. What per cent. of evaporation must take place from an a% solution to produce a b% solution? ($b>a$).

67. A physician having a 6% solution of a certain kind of medicine wishes to dilute it to a $3\frac{1}{2}$% solution. What per cent. of water must be added?

68. A druggist has a 95% solution. What must he do to change it to an 80% solution required in a prescription?

69. From a mixture of sugar and water of 30% strength, $\frac{1}{4}$ of it evaporated (removing water only). What per cent. of strength was the remainder?

70. From a mixture of medicine and alcohol of 28% strength, $\frac{1}{2}$ was evaporated (removing alcohol only). What per cent. of strength was the remaining medicine?

71. Solve the following equations for r:

(1) $0.25r-0.125r=0.75$

(2) $\dfrac{r+2.1}{r+0.1}+\dfrac{r}{r+0.7}=2.275$

(3) $\dfrac{\frac{r}{3}+\frac{1}{2}}{5}-\dfrac{r-\frac{r}{3}}{2}=-\frac{3}{2}$

(4) $\dfrac{a+r}{a+c}-\dfrac{a-r}{c-a}=\dfrac{a-r}{c-a}$

(5) $5r-13=\dfrac{2r-5}{4}+\dfrac{r+4}{4}$

(6) $\dfrac{5r^2-3r+12}{7}=10\dfrac{(3r+1)(r-10)}{42}$

(7) $2r-\dfrac{6r^2-2r+1}{6}+\dfrac{2r^2-3r}{2}=-1$

(8) $15r-\dfrac{6r+1}{2}-\dfrac{r-1}{3}=-6.$

72. A box of oranges was bought at the rate of 15 cents a dozen. Five dozen were given away and the remainder sold at the rate of 2 for 5 cents. If this gave a profit of 30 cents on the box, how many were there in the box?

73. A father engaged his son to work 20 days on the following conditions: For each day he worked he was to receive \$2, and for each day he was idle he was to forfeit \$1. At the end of 20 days he received \$34. How many days was he idle?

74. A man engaged to work on the following conditions: For each day he worked he was to receive b dollars, and for each day he was idle he was to forfeit c dollars. At the end of a days he received d dollars. How many days was he idle?

75. The tire of the fore-wheel of a carriage is 9 ft., that of the hind-wheel 12 ft. What distance will the carriage have passed over when the fore-wheels have made 5 more revolutions than the hind-wheels?

76. The tire of the fore-wheel of a carriage is a ft., that of the hind-wheel b feet. What distance will the carriage have passed over when the fore-wheels have made n more revolutions than the hind-wheels?

77. A man spends one ath part of his income for food, one bth part for rent, one cth part for clothing, one dth part for furniture, and he saves e dollars. How much is his income?

Exercise XXXIII

158. Solve the following equations:

1. $A=\frac{b+b'}{2}\cdot h$, for h; for b; or b'
2. $i=\frac{prt}{100}$ for p; for r; for t; for pr; for rt
3. $V=\frac{1}{3}bh$ for b; for h
4. $C=\frac{5}{9}(F-32)$ for F
5. $pd=PD$, for p; for d; for P; for D
6. $C=\frac{E}{R+r}$ for E; for R; for r
7. $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$ for F; for f_1; for f_2
8. $C=\frac{1}{C_1}+\frac{1}{C_2}$ for C_1; for C_2
9. $\frac{1}{8}WL=\frac{SI}{c}$ for W; for S; for $\frac{I}{c}$
10. $v=v_0\left(1+\frac{t}{273}\right)$ for v_0; for t.

Exercise XXXIV

159. Solve the following exercises for x:

1. $ax-bx=-(b-a)$
2. $cx+ax=a^2+c^2+2ac$
3. $3x-ax=a^2+9-6a$
4. $m^2x-n^2x=n+m$
5. $s^2x+r^2x-2rsx=r^2-s^2$
6. $\frac{x}{a}-\frac{x}{b}=\frac{b^2-a^2}{ab}$
7. $\frac{x}{4b}-\frac{x}{3b}=\frac{6a-8b}{12ab}$
8. $\frac{x}{a}-x=1+\frac{1}{a^2}-\frac{2}{a}$
9. $(2x-1)(3-x)=-2(x^2-9)$
10. $x^2-c^2=3c^2-2cx+x^2$
11. $x^3-2ax+c=a+3ax+x^3$
12. $a^2+ax+cx-ac=0$
13. $3rx-2cm=3cx-2rm$
14. $\frac{2x-l}{6x+s}=\frac{x-2s}{3x+b}$
15. $\frac{2x-a}{c}-\frac{x-a}{d}=\frac{a}{c}$
16. $\frac{cx-d}{dx}+\frac{dx-c}{cx}=2+\frac{c-d}{cdx}$
17. $\frac{2x+a}{c}-\frac{x+a}{d}=-\frac{a}{c}$
18. $\frac{x^2-d}{cx}+\frac{d+x}{c}=\frac{2x}{c}-\frac{d}{x}$
19. $(x+m)^2+(n+s)^2=(x-m)^2+(n-s)^2$
20. $\frac{r-s}{x-t}+\frac{s-t}{x-r}=\frac{r-t}{x}$
21. $\frac{r-s}{x-t}-\frac{s-t}{x-r}=\frac{(s-t)(t-r)}{(x-t)(x-r)}$
22. $\frac{s-r}{x+t}-\frac{t-s}{r+x}=\frac{(s-t)(t-r)}{(x+t)(x+r)}$
23. $\frac{cx}{x+d}+\frac{dx}{x+c}=c+d$
24. $\frac{(2x-5r)^2}{(2x-5s)^2}=\frac{x-5r}{x-5s}$
25. $\frac{1}{x-p}+\frac{1}{x-q}=\frac{2x-p-q}{x(x-p-q)}$
26. $5.8x+3.69=3.96+2.8x$
27. $.374x-.53+1.2x+.06=.8+1.32x$
28. $.3(1.5x-.8)=.6(5.1+.2x)$

29. $.05(20x-3.2)=.8(4x+.12)-11.256$

30. $1.4x-1.61-\frac{.21x+.012}{.8}=1.3x$

31. $\frac{1-2x}{.25}-\frac{2x-.5}{12.5}+\frac{2x-\frac{1}{3}}{5}=\frac{6.35-.5x}{3}$

32. $\frac{.4x+.39}{.7}-\frac{.2x-.66}{.9}=\frac{.08x+.38}{.2}$.

Summary

Algebraic expressions may be *pictured*, or *graphed*.

An equation symbolizes the question:

What value, or what values, if any, of the unknown, are there which, substituted for the unknown, makes the first member identical with the second member?

The **identical equation**, or the **identity**, is an equation that is true for any, or all, values of its letters.

The distinguishing sign of identity is $\equiv$.

The **conditional equation** is an equation that is true for one value, or for a definite number of values, of its letters.

The *left side* of an equation is called the **left member**, or the **first member**, and the *right side* the **right member**, or the **second member.**

An algebraic problem is stated

I. *By denoting the unknown by a letter and then translating the verbal statement of the number relations into a symbolic statement in equation form.*

II. *By expressing some number in the problem in two different forms and writing the two expressions equal to one another.*

To solve an equation in one unknown means to find the number, or the numbers, which, substituted for the unknown (the variable), reduces both members to the same number.

Any number that fulfills this condition is a *root of the equation.*

To **check the solution of an equation** means to show that the result is the root of the equation.

To **check the solution of a problem** means to show that the result answers the conditions stated in the problem.

CHAPTER XI

LINEAR EQUATIONS CONTAINING TWO OR MORE UNKNOWN NUMBERS. GRAPHIC SOLUTION OF EQUATIONS AND PROBLEMS

Indeterminate Linear Equations

160. The meaning of the solution of equations and of problems is made clearer by means of graphic methods.

1. How many 8 and 10 candle-power bulbs are necessary to obtain an 82 candle-power light? (More than one solution.)

Let x and y denote the number of 8's and 10's respectively.

Then $8x+10y=82$, or $4x+5y=41$

and $$y=\frac{41-4x}{5}.$$

From which, if $x=-1$ then $y=9$ and if $x=14$ then, $y=-3$.

These solutions are the *co-ordinates* of two points on the graph; viz., A $(-1, 9)$ and B $(14, -3)$. (Fig. 226.)

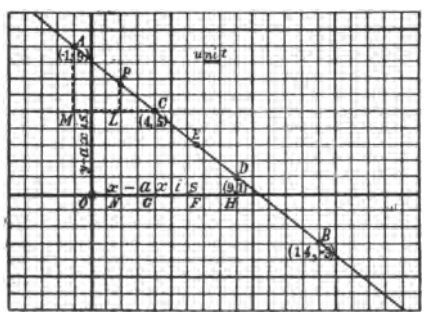


FIG. 226

Since x and y must be positive integers to satisfy the problem and since there are on the graph only two points whose

co-ordinates are positive integers, namely, C (4, 5) and D (9, 1), then $x=4$, $y=5$; and $x=9$, $y=1$ are *possible* solutions.

What are the co-ordinates of point E? Do the co-ordinates of point E check in the equation? In the problem?

What good reason is there for always checking in the problem?

What is the *origin?* What are the co-ordinates of the origin?

Find the co-ordinates of the point P. (Find L P by proportion.)

Where in respect to the x- and y-axes do you look for points having positive integral co-ordinates?

What important relation is there between the equation and the co-ordinates of all points on the graph of the equation?

Why may $8x+10y=82$ be called an *indeterminate* equation? A *simple* equation? A *linear* equation?

Why were only two points necessary to construct the graph?

Why should the two points which determine the graph be chosen some distance apart and have integral co-ordinates?

2. A contractor has cornice stones 3 and 4 feet in length. How many of each may he use, without cutting, to lay the cornice of a wall 46 ft. long? Give all the positive integral solutions of the equation.

3. In how many ways can a merchant make an even exchange of hats at \$4.00 apiece for gloves at \$3.00 a pair? Give the least positive integral solution of the equation.

4. At how many feet from the fulcrum will a 75-pound weight on one side balance a 45-pound weight on the other? Give the least positive integral solution.

5. Weights of 12 and 8 pounds are placed at opposite ends of a lever resting on a fulcrum. By moving the 12-lb. weight 3 ft. toward the fulcrum, the weights balance. How far is

the fulcrum from each end of the lever and how long is the lever? (Several solutions.)

Let x and y denote the required distances. Then $x-3$ will denote the number of feet the 12-lb. weight is from the fulcrum.

Interpret each solution.

What is the least positive solution of the equation?

What form have the equations of problems 3 and 4?

Which term is missing?

Their graphs have what point in common?

What are the co-ordinates of this point?

Do the graphs of all the equations of this form, $ax=by$, pass through the origin?

Do all of them have the solution $x=0$ and $y=0$?

Test your answers with the following equations: $4x=7y$, $3x+2y=0$, and $4x-5y=0$.

A System of Two Linear Equations

161. Some problems lead to two linear equations in two unknowns.

1. Weights of 12 lb. and 8 lb. are placed at opposite ends of a lever resting on a fulcrum.

(*a*) If the 12-lb. weight is moved 3 ft. toward the fulcrum they balance.

(*b*) Changing the weights from end to end, and moving the 12-lb. weight $1\frac{1}{3}$ ft. toward the fulcrum, they balance again.

How far is the fulcrum from each end of the lever? (See problem 5, p. 235.)

Letting x and y denote the required distances, then from

Condition (*a*): $12(x-3)=8y$; or $3x-2y=9$ (1)

Condition (*b*): $8x=12(y-1\frac{1}{3})$, and $2x-3y=-4$. (2)

Construct the graphs of (1) and (2) on the same axes.

Find the co-ordinates of several points on the graph of equation (1). Do they satisfy equation (1)? Condition (*a*)?

Consider similarly co-ordinates of points on the graph of equation (2).

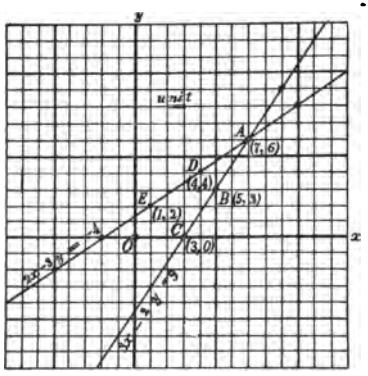


FIG. 227

What must be true of the co-ordinates of a point common to both graphs?

What is evidently the solution of the problem? Check in both equations and in the problem.

2. Find two integral numbers such that, (*a*) 4 times one diminished by 3 times the other is equal to -22, and (*b*) the sum of the numbers is equal to -9.

3. Find two integral numbers such that, (*a*) their sum is equal to -9, and (*b*) 6 times the first diminished by 5 times the other is equal to 45.

4. Find two integral numbers such that 6 times the first diminished by 5 times the other is equal to 45, and the sum of 3 times the first and 2 times the second is equal to 9.

5. Find two integral numbers such that the sum of 3 times the first and 2 times the other is equal to 9, and 4 times the first diminished by 3 times the second is equal to -22.

162. It was seen that each of the preceding problems

produced a pair of linear equations having one and only one *common* solution.

Such equations are commonly called a **system of two linear equations**.

Exercise XXXV

163. Solve the following systems by the graphic method, and check:

(1) $x+2y=17$
$3x-y=2$

(2) $7x+3y=-36$
$5x-2y=-5$

(3) $9x-6y=36$
$13x+5y=11$

(4) $2x-3y=4$
$4x+5y=30$

(5) $8x+5y=44$
$2x-y=2$

(6) $11x+7y=40$
$3x-5y=4$.

A Pair of Contradictory Linear Equations

164. Find two numbers such that, (*a*) their sum is 10, and (*b*) 3 times the first increased by 3 times the second is equal to 15.

Letting x and y denote the required numbers, then from

Condition (*a*) $x+y=10$
Condition (*b*) $3x+3y=15$, or $x+y=5$
That is, $x+y=10$
And, *at the same time*, $x+y=5$.

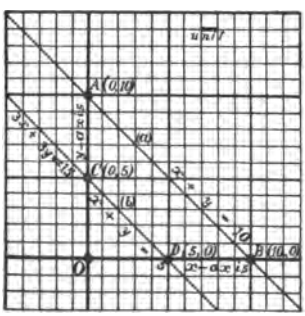


Fig. 228

Construct the graphs of the last two equations on the same axes.

Are O B and O A divided proportionally by D C? Why?

Is C D parallel to A B? Why?

What does this tell about the problem? The equations?

What two contradictory statements are expressed in the equations themselves?

The equations of such a pair are called *inconsistent*, or *contradictory* equations.

A Pair of Equivalent Linear Equations

165. Find two numbers such that, (*a*) their sum is 8, and (*b*) $\frac{1}{3}$ of the first increased by $\frac{1}{3}$ of the other is equal to $2\frac{2}{3}$.

Letting x and y denote the required numbers, then from

Condition (*a*) $x+y=8$

Condition (*b*) $\frac{1}{3}x+\frac{1}{3}y=2\frac{2}{3}$, or $x+y=8$.

The graphs evidently coincide. Hence the problem and its equations have an unlimited number of solutions. Give several of these solutions. The equations of such a pair are called *equivalent*, or *dependent* equations.

Exercise XXXVI

166. Are the following pairs of equations contradictory, or equivalent, and are their graphs parallel, or coincident? Give a reason for each answer.

1. $3x+\frac{y}{4}=6$
   $4x+\frac{y}{3}=8$
2. $x+\frac{2}{7}y=2$
   $\frac{x}{2}+\frac{1}{7}y=1$
3. $7x-8=4y-2x$
   $18x-8y=16$
4. $3x-2y=14$
   $9x-6y=36$
5. $3x+2y-7-x=12-3y$
   $2x+5y=20$
6. $3x+4y=12$
   $6x+8y=14.$

The Solution of a System of Two Linear Equations by Algebraic Methods

167. (*a*) John sets out on a walking trip and travels at a uniform rate for 5 hours when he meets with an accident. He continues, however, at a slower pace, and 3 hours later reaches

a point 26 miles from home. (*b*) If he had turned back at the time of the accident he would have reached in 3 hours a point 14 miles from home. What was his rate of speed both before and after the accident?

Letting x and y denote the rate before and after the accident respectively,

Condition (*a*) $5x+3y=26$ (1)
Condition (*b*) $5x-3y=14$ (2)

(Add. Ax.) $10x=40$ (Sub. Ax.) $6y=12$
(Div. Ax.) $x=4$ (Div. Ax.) $y=2$

Check: $5(4)+3(2)=26$, and $5(4)-3(2)=14$.
Check in the problem also.

The equation $10x=40$ was obtained from equations (1) and (2) by what operation? What terms were canceled (eliminated) from equations (1) and (2) by this operation?

What is true of the coefficients of the y-terms which makes possible the elimination of these terms *by addition?*

The equation $6y=12$ was obtained from equations (1) and (2) by what operation? What terms were eliminated from equations (1) and (2) by this operation? What is true of the coefficients of the y-terms which makes possible the elimination of these terms *by subtraction?*

168. What must be done when the coefficients of x and y are not alike, as in the following example?

Solve: $\left.\begin{array}{ll} 5x+3y=26 & (1) \\ 4x-7y=2 & (2) \end{array}\right\}$ (*a*)

If equation (1) is multiplied by the coefficient of x in equation (2), and equation (2) by the coefficient of x in equation (1), what is the coefficient of x in each of the resulting equations?

What operation will then eliminate the x-terms from the resulting equations?

How may the y-terms be eliminated from equations (1) and (2)?

Show from the following whether your answers are correct:

$$\left.\begin{array}{rl} 7\times(1), & 35x+21y=182 \\ 3\times(2), & 12x-21y=6 \end{array}\right\}(b) \qquad \left.\begin{array}{rl} 4\times(1), & 20x+12y=104 \\ 5\times(2), & 20x-35y=10 \end{array}\right\}(c)$$

$$\begin{array}{rl} 47x & =188 \\ x & =4 \end{array} \qquad \begin{array}{r} 47y=94 \\ y=2. \end{array}$$

Having found the value of x as shown above, the value of y might have been found by substituting the value of x in one of the given equations, thus

$$\begin{array}{rl} 5x+3y=26 & \text{Check in (1) } 5\cdot 4+3\cdot 2=26 \\ 5\cdot 4+3y=26 & \\ 3y=\ 6 & \text{Check in (2) } 4\cdot 4-7\cdot 2=\ 2. \\ y=\ 3 & \end{array}$$

Do the values $x=4$ and $y=2$ satisfy system (a)? System (b)? System (c)? (a), (b), and (c) are called *equivalent systems.*

How were systems (b) and (c) obtained from system (a)?

From your answers to the questions above, make a rule for solving a system of two linear equations.

169. This method of obtaining from two equations a single equation with one unknown is called **elimination by addition and subtraction.**

Exercise XXXVII

1. Tell how to obtain from each of the following an equivalent system having the coefficients of the first unknown the same; of the second unknown the same.

(1) $4x+3y=13$
$3x+2y=9$

(2) $5x+4y=22$
$3x-7y=-15$

(3) $3x-5y=51$
$2x+7y=3$

(4) $2t-7s=58$
$-9t+4s=69$

(5) $7x+5y=17$
$11x-3y=5$

(6) $6a-4b=2$
$5a+7b=43$

(7) $-11x+9y=16$
$4x+8y=28$

(8) $3u-\ 2v=4$
$-7u+13v=-1$

(9) $13x+10y=59$
$11x-\ 9y=15.$

2. Solve the systems of problem 1 and check.

170. Solve

$$\begin{cases} 3b-5+\dfrac{2a-b}{4}=\dfrac{a+7}{5} & (1) \\[2ex] \dfrac{2a}{3}-\tfrac{1}{2}+5a-18=\dfrac{7-5b}{2} & (2) \end{cases}$$

Removing denominators

$$20\times(1),\quad 6a+55b=128 \qquad (3)$$
$$6\times(2),\quad 34a+15b=132. \qquad (4)$$

When the coefficients of an unknown have a common factor, the work of eliminating that unknown can be made easier by multiplying each equation by the remaining factor of the coefficient of that unknown in the other equation after removing the common factor.

To eliminate the a-terms from equations (3) and (4), above, what are the smallest integral multipliers? To eliminate the b-terms, what are the smallest integral multipliers?

Show whether your answers are correct by comparing them with the following:

$$\begin{aligned} 17\times(3),\quad 102a+935b&=2{,}176 \\ 3\times(4),\quad 102a+\ 45b&=396 \\ 890b&=1{,}780 \\ b&=2 \end{aligned} \qquad \begin{aligned} 3\times(3),\quad 18a+165b&=384 \\ 11\times(4),\quad 374a+165b&=1{,}452 \\ 356a&=1{,}068 \\ a&=3. \end{aligned}$$

Supplement the preceding rule, §168, so as to include systems like the foregoing.

Exercise XXXVIII

171. Tell how to obtain from each of the following in the first column an equivalent system having the coefficients of the first unknown the same; of the second unknown the same.

1. $12x+15y=66$
   $16x-25y=-2$
2. $8h-21y=33$
   $6h+35y=177$
3. $33A-28B=38$
   $22A+35B=79$
4. $66m+55n=299$
   $77m-15n=201$

5. $21u+69v=111$
$14u-26v=2$

6. $51d+76w=203$
$34d-38w=-42$

7. $(2u+3):5=(3v+5):7$
$7u:4v \quad =77:40$

8. $3x+\dfrac{2x-y}{4}=5+\dfrac{x+7}{5}$
$5x+18+\dfrac{2x}{3}=\frac{1}{2}+\dfrac{7-5y}{2}$

9. $(2R+3)(3r-7)=(3R-5)(2r+13)+29$
$(5R+7)(2r+9)=R(10r+25)+111$

10. $\frac{1}{7}u-\frac{2}{3}(w+1)=\frac{1}{3}(u-1)-9$
$\frac{1}{6}(w+1)-u=\frac{1}{2}(w-1)-10$

11. $\dfrac{x+y}{2}-\dfrac{x-y}{3}=\frac{7}{6}$
$\dfrac{x+y}{3}+\dfrac{x-y}{4}=\frac{5}{4}$

12. $a+\frac{1}{2}(3a-b-1)=\frac{1}{4}+\frac{3}{4}(b-1)$
$\frac{1}{10}(7b+24)=\frac{1}{5}(4a+3b).$

13. Solve each of the 12 systems above.

172. Elimination by Substitution and by Comparison

ELIMINATION BY SUBSTITUTION

Solve: $5x-7y=1$ (1)

$3x+4y=17$ (2)

From (1) $x=\dfrac{1+7y}{5}$ (3)

Substitute (3) in (2) $3\dfrac{(1+7y)}{5}+4y=17$ (4)

(Mult. Ax.) $3+21y+20y=85$ (5)

(Sub. Ax.) $41y=82$ (6)

(Div. Ax.) $y=2$ (7)

Substituting $y=2$ in (3) $x=3.$ (8)

Elimination by Comparison

Solve:	$5x-7y=1$	(1)
	$3x+4y=17$	(2)
From (1)	$x=\frac{1+7y}{5}$	(3)
From (2)	$x=\frac{17-4y}{3}$	(4)
(Comp. Ax.*)	$\frac{1+7y}{5}=\frac{17-4y}{3}$	(5)
(Mult. Ax.)	$3+21y=85-20y$	(6)
(Add. and Sub. Ax.)	$41y=82$	(7)
(Div. Ax.)	$y=2$	(8)
Substituting	$x=3.$	(9)

Exercise XXXIX

173. Solve each of the following by the method of *substitution* and check by the method of *comparison:*

1. $4x+5y=14$
   $3x-2y=-1$
2. $\frac{1}{2}x+\frac{2}{5}y=9$
   $\frac{1}{3}x+\frac{1}{2}y=7$
3. $7x-2y=8$
   $3x+4y=18$
4. $2m+11k=50$
   $11m+2k=41$
5. $9R-2r=44$
   $6R-\ r=31$
6. $13u-6v=22$
   $4u+9v=61$
7. $7w-2z=46$
   $w+\ z=13$
8. $\frac{2}{7}h+\frac{5}{2}k=33$
   $\frac{3}{4}h-\frac{2}{5}k=17.$

174. If the time required to do a piece of work is:

(1) 10 days, what part of it is done in 1 day? In 3 days? In 10 days?

(2) x days, what part of it is done in 1 day? In 3 days? In x days?

* *Comparison Axiom:* Numbers that are equal to the same number are equal numbers.

(3) y days, what part of it is done in 1 day? In 3 days? In y days?

If A worked 3 days on a piece of work and B 2 days, they can do $\frac{14}{15}$ of it. But if A works 2 days and B 3 days, they can do $\frac{9}{10}$ of it. In how many days can each one do it, working alone?

Letting x and y denote the number of days required by A and B, respectively, then $\frac{1}{x}$ and $\frac{1}{y}$ will denote the parts A and B, respectively, can do in one day.

$$\text{Whence,} \quad \frac{3}{x}+\frac{2}{y}=\frac{14}{15} \qquad (1)$$

$$\text{And,} \quad \frac{2}{x}+\frac{3}{y}=\frac{9}{10}. \qquad (2)$$

These equations are not linear in x and y, but are linear in $\frac{1}{x}$ and $\frac{1}{y}$ and should be solved for $\frac{1}{x}$ and $\frac{1}{y}$. They should not be cleared of fractions. From the values of $\frac{1}{x}$ and $\frac{1}{y}$, x and y can easily be found.

What operation would eliminate the y-terms if their numerators were the same? Solve the equations.

Exercise XL

175. Solve the following without clearing of fractions, and check:

1. $\frac{10}{x}-\frac{9}{y}=\frac{1}{20}$
   $\frac{4}{x}+\frac{3}{y}=\frac{1}{6}$

2. $\frac{7}{x}+\frac{4}{y}=\frac{11}{30}$
   $\frac{5}{x}-\frac{6}{y}=-\frac{3}{28}$

3. $\frac{5}{x}+\frac{6}{y}=\frac{24}{143}$
   $\frac{13}{x}-\frac{11}{y}=\frac{1}{30}$

4. $\frac{7}{x}+\frac{9}{y}=\frac{22}{105}$
   $\frac{15}{x}-\frac{21}{y}=-\frac{4}{21}$

5. $\frac{11}{x}-\frac{2}{y}=\frac{1}{6}$
   $\frac{2}{x}+\frac{3}{y}=\frac{13}{11}$

6. $\frac{6}{x}+\frac{3}{y}=\frac{13}{60}$
   $\frac{9}{x}+\frac{20}{y}=\frac{7}{12}$.

Systems of Linear Equations in Two Unknowns Having Literal Coefficients

176. Solve for x and y: $a^2x+b^2y=a+b$ (1)

$abx-aby=b-a$ (2)

$a\times(1)$ $a^3x+ab^2y=a^2+ab$ (3)

$b\times(2)$ $ab^2x-ab^2y=b^2-ab$ (4)

(Add. Ax.) $(a^3+ab^2)x=a^2+b^2$ (5)

$$x=\frac{a^2+b^2}{a^3+ab^2}=\frac{a^2+b^2}{a(a^2+b^2)}=\frac{1}{a}.$$

Find the value of y, first by eliminating the x-terms from equations (1) and (2); and then by substituting $x=\frac{1}{a}$ in equation (2). Check your results.

Exercise XLI

1. Tell how to solve each of the following; then solve and check:

(1) $x+y=1$
$ax-by=0$

(2) $cx+ny=1$
$ax-by=0$

(3) $ax+by=h$
$bx+ay=k$

(4) $cx+dy=2cd$
$bx-cy=d-c$

(5) $ax+by=2ab$
$2bx+3ay=2b^2+3a^2$

(6) $a_1x+b_1y=c_1$
$a_2x+b_2y=c_2$

(7) $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$
$\frac{1}{x}-\frac{1}{y}=\frac{1}{k}$

(8) $\frac{a}{x}+\frac{b}{y}=c$
$\frac{d}{x}+\frac{e}{y}=f$

(9) $\frac{m}{x}+\frac{n}{y}=k$
$\frac{n}{x}+\frac{m}{y}=h$

(10) $\frac{2a}{x}-\frac{3b}{y}=1$

$\frac{2b}{x}+\frac{3a}{y}=1$

(11) $\frac{x+a}{y+b}=\frac{a+c}{b+d}$

$\frac{x+1}{y+1}=\frac{c+1}{d+1}$

(12) $\frac{x}{y}=\frac{a+b}{a-b}$

$\frac{x+c}{y-c}=\frac{a+b+c}{a-b-c}$

(13) $\frac{x+1}{y+1}=\frac{h+1+k}{h+1-k}$

$x-y=2k$

(14) $\frac{x-b}{y-b}=\frac{b-c}{b+c}$

$\frac{x+b}{y+b}=\frac{3b-c}{3b+c}$

(15) $\frac{x}{a+b}+\frac{y}{a-b}=\frac{1}{a-b}$

$\frac{x}{a+b}-\frac{y}{a-b}=\frac{1}{a+b}$.

Systems of Three Linear Equations

177. The sum of 3 times the first, 5 times the second, and 3 times the third of three numbers is equal to 22. The sum of 5 times the first, and 3 times the second, diminished by 4 times the third, is equal to -1. If from the sum of 4 times the first and twice the second, 5 times the third is subtracted, the remainder is -7. What are the numbers?

Letting x, y, and z denote the first, second, and third numbers respectively, the equations are:

$$3x+5y+3z=22 \qquad (1)$$
$$5x+3y-4z=-1 \qquad (2)$$
$$4x+2y-5z=-7. \qquad (3)$$

Eliminating z by combining (1) and (2), and again by combining (1) and (3), or (2) and (3), what two unknowns will the two resulting equations contain? Can these equations be solved for x and y?

Having found x and y, how is z found? If we eliminate first z and then y, will the resulting equations contain the same two unknowns? What two unknowns will they con-

tain? Can such a pair of equations have one and only one solution?

Eliminate z from (1) and (2) thus:

$4\times(1)$	$12x+20y+12z=88$	(4)
$3\times(2)$	$15x+9y-12z=-3$	(5)
(Add. Ax.)	$27x+29y=85$	(6)

Next, eliminate z from (1) and (3) thus:

$5\times(1)$	$15x+25y+15z=110$	(7)
$3\times(3)$	$12x+6y-15z=-21$	(8)
	$27x+31y=89$	(9)

Solving (6) and (9), $x=1$ and $y=2$

Substituting in (2), $5\cdot 1+3\cdot 2-4z=-1$

$z=3.$

Check in (1):

$3\cdot 1+5\cdot 2+3\cdot 3=22$; i. e., $22=22$.

Check in (2):

$5\cdot 1+3\cdot 2-4\cdot 3=-1$; i. e., $-1=-1$.

Check in (3):

$4\cdot 1+2\cdot 2-5\cdot 3=-7$; i. e., $-7=-7$.

Write out a rule for solving a system of three linear equations.

Compare your rule with the following:

Make two different pairs of equations out of the three equations, and eliminate the same unknown from both of these pairs. This gives two equations in two unknowns. Solve the two resulting equations as a system of two linear equations.

Exercise XLII

178. Tell how to obtain from each system a single equation containing only the first unknown; the second unknown; the third unknown.

1. $3x+y-2z=-1$
$-4x+2y+3z=9$
$5x+3y-2z=5$

2. $2u+2v+w=9$
$u+3v+2w=13$
$3u-3v+4w=9$

3. $\frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=\frac{23}{12}$
$\frac{1}{3}x+\frac{1}{4}y+\frac{1}{2}z=\frac{7}{8}$
$\frac{1}{4}x+\frac{1}{2}y+\frac{1}{3}z=\frac{9}{4}$

4. $a+3b+5c=21$
$3a-2b-4c=22$
$4a-3b-6c=28$

5. $5x+3y=45$
$7y-2z=27$
$3z-4x=-12$

6. $4u-v=50$
$5v-2w=40$
$6w-u=15$

7. $3x+4y+6z=2x+6y+5z=4x+2y+9z=68$

8. $\frac{2}{x}+\frac{3}{y}-\frac{1}{z}=\frac{3}{4}$
$\frac{5}{x}-\frac{2}{y}+\frac{1}{z}=\frac{17}{20}$
$\frac{3}{x}+\frac{1}{y}+\frac{2}{z}=\frac{4}{5}$

9. $\frac{3}{v}-\frac{2}{u}=\frac{2}{5}$
$\frac{7}{u}-\frac{4}{w}=\frac{1}{2}$
$\frac{10}{w}-\frac{2}{v}=\frac{1}{16}$

10. $\frac{7}{x}+\frac{5}{y}-\frac{2}{z}=\frac{53}{6}$
$\frac{5}{x}-\frac{2}{y}+\frac{3}{z}=5$
$\frac{2}{x}+\frac{9}{y}-\frac{5}{z}=\frac{29}{6}$

11. $\frac{x+1}{y+1}=2$
$\frac{y+2}{z+1}=1$
$\frac{z+3}{x+1}=\frac{5}{4}$.

12. Solve the above systems and check.

Problems Involving Two Unknown Numbers

179. The student should make a practice of re-reading each problem until the conditions imposed upon the unknown numbers are fully understood.

1. A street railway company receives a certain sum for each cash fare and a different sum for each transfer. On one trip the conductor of a car collects 13 cash fares and 18 transfers amounting to $1.10 for the company. On the return

trip there were 7 cash fares and 24 transfers and the amount for the company was \$0.95. What does the company receive for a cash fare? For a transfer?

2. Three tons of hard coal and two tons of soft coal cost \$32. The price remaining the same, 2 tons of hard coal and 6 tons of soft coal cost \$43.50. What were the prices per ton of the two kinds of coal?

3. One of the base angles, x, of an isosceles triangle is equal to twice the vertex-angle, y. Find all the angles of the triangle.

4. The difference of the acute angles of a right triangle is 36°. Find the number of degrees in each acute angle.

5. The difference of the adjacent angles of a parallelogram is 20°. Find the values of all the angles of the parallelogram.

6. The consecutive angles a and b of a parallelogram are so related that $3a-b=30°$. Find the values of both of the angles.

7. Three times one of two adjacent sides of a parallelogram exceeds twice the other by 45, and the perimeter of the parallelogram is 80. Find the length of the sides.

8. One dimension of a rectangle is 5 and one dimension of another is 3. The sum of the areas is 65 and the difference 35. Find the dimensions of the rectangles.

9. Two trains pass each other going in the same direction, with a relative speed of 10 miles an hour. Going in opposite directions, they would pass with a relative speed of 70 miles an hour. Find the speeds of the trains.

10. The sum of the areas of two rectangles of dimensions 5 and x, and 3 and y, is 49, and the area of a rectangle of dimensions 3 and x exceeds the area of the rectangle of dimensions 5 and y by 9. Find the dimensions x and y.

11. The following advertisement was published by an electric company:

The price of electricity has dropped since 1902 by a certain number of cents.

The number of units of electricity this amount of money will buy has increased by a certain number. The sum of these two numbers is 59 and the difference is 19. Find the numbers.

Solve the problem.

12. A man invests part of $3,200 at 6 per cent. and the rest at 5 per cent. If his annual income is $180, how much did he invest at each rate?

13. A man gained 8 per cent. on one investment and lost 3 per cent. on another. If the money invested amounted to $22,000 and the net gain was $440, what was the amount of each investment?

14. Two investments, one at $3\frac{1}{3}$ per cent. and the other at $5\frac{1}{2}$ per cent., yield annually $150. If the first had been at $8\frac{3}{4}$ per cent. and the other at $3\frac{1}{2}$ per cent., the annual income would have been $175. What was the amount of each investment?

15. The areas of two triangles, having equal bases, are 72 sq. in. and 60 sq. in. Twice the altitude of the first plus 3 times the altitude of the second is equal to 54 inches. Find the altitudes.

16. The areas of two circles are to each other as 4 is to 36. One-half the radius of the first, plus $\frac{1}{7}$ of the radius of the second, is equal to $6\frac{1}{2}$. Find the radii of the circles and their areas.

The formula for the area of a circle is: $A = \pi r^2$. Find the ratio of the radii.

17. The ratio of the circumferences of two circles is 2, and 6 times the radius of the first minus 4 times the radius

of the second is equal to 14. Find the radii of the circles and their circumferences.

The formula for the circumference of a circle is: $c=2\pi r$.

18. The altitude of a trapezoid is 8 and the area is 56. If the lower base is increased by a length equal to the upper base, the area is 72. Find the bases of the trapezoid.

19. The lower base of a trapezoid is 24 and the area is 150. If a length equal to $\frac{1}{6}$ of the lower base is added to the upper base the area is 170. Find the altitude and upper base of the trapezoid.

Exercise XLIII

20. Solve the following:

(1) $\begin{cases} s+6p=4s+5p \\ (s+4)-(5p-s)-s=0 \end{cases}$

(2) $\begin{cases} \frac{4}{3}R-\frac{1}{4}r=5 \\ \frac{5}{2}R-\frac{2}{3}r=7 \end{cases}$

(3) $\begin{cases} mx-ky=mR \\ kx+my=kR \text{ (for } x \text{ and } y) \end{cases}$

(4) $\begin{cases} \dfrac{6-H}{2}-\dfrac{R+5}{5}=-12 \\ \frac{1}{3}(12+H)-\frac{1}{9}(R-4)=4 \end{cases}$

(5) $\begin{cases} \dfrac{3}{R}-\dfrac{2}{r}=\frac{1}{7} \\ \dfrac{5}{R}+\dfrac{7}{r}=\frac{5}{8} \end{cases}$

(6) $\begin{cases} -\dfrac{20-2K}{3}+\dfrac{5K}{6}=\dfrac{H}{6}+\dfrac{4K-19}{3} \\ 5-\dfrac{2K+21}{3}=-\dfrac{H+5K}{6} \end{cases}$

$$(7)\quad \begin{cases} 2p+2s+3v=4 \\ 3p+4s+6v=7 \\ p+2s+6v=4 \end{cases}$$

$$(8)\quad \begin{cases} H+K=2c \\ K+R=2a \text{ (for } H, K, \text{ and } R) \\ R+H=2b \end{cases}$$

(9) Construct the graphs of (2) and (6).

21. In the right triangle A B C, C D is perpendicular to the hypotenuse A B. Find m and n.

The segments of the hypotenuse are to each other as the squares of the two sides *adjacent to the segments.*

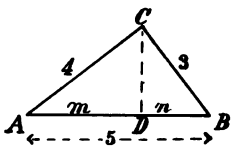


FIG. 229

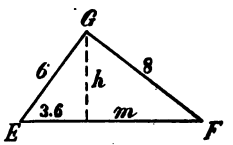


FIG. 230

22. In the right triangle E F G, h is perpendicular to the hypotenuse E F. Find m and h.

Apply the principles of similar triangles.

23. In Fig. 231 x has the same value throughout. The same is true of y. Find x and y from the relations indicated

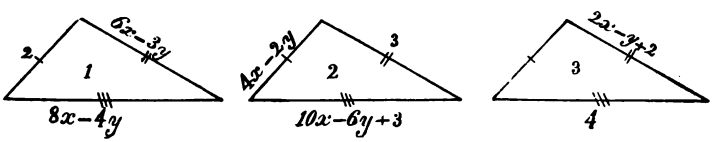


FIG. 231

(1) in triangles 1 and 2; (2) in triangles 1 and 3; (3) in triangles 2 and 3.

24. The perimeter of the first triangle, Fig. 232, is 288. Find x and y from the relations indicated in the triangles.

25. The corresponding altitudes of two similar triangles are represented by the expressions $3x+4y-7$ and $2x-y$

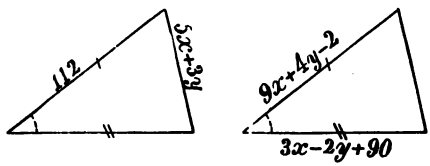


FIG. 232

respectively, and the corresponding bases by the expressions $5x-2y+1$ and $4x-2y$ respectively. The perimeter of the first triangle is 4 and of the second is 6. Find he values of x and y.

The perimeters of two similar triangles are to each other as the corresponding sides or altitudes.

26. In Fig. 233 D E is parallel to B C. The sides A B and A C are represented by the expressions $4x+3y+1$ and $6x+4y+2$ respectively. Find x and y.

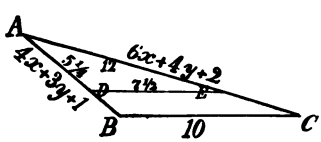


FIG. 233

27. A number of two digits is equal to 9 less than 7 times the sum of the digits. If the digits are reversed the number is 18 less than the original number. What is the original number?

28. Three times the reciprocal of the first of two numbers and 4 times the reciprocal of the second are together equal to 5. Seven times the reciprocal of the first less 6 times the reciprocal of the second is equal to 4. What are the numbers?

29. If the numerator of a certain fraction is increased by $2a$ and the denominator by $3b$, the resulting fraction is equal to $\frac{a}{b}$. If the numerator and denominator are decreased by $12a$ and $6b$ respectively, the resulting fraction is equal to $\frac{2a}{b}$. What is the original fraction?

30. A and B, working together, build a fence in 4 days. They can also build it if A works 3 days and B 6 days. In how many days can each alone build the fence?

31. A boat crew rows 4 miles down stream in 20 minutes and the same distance up stream in 35 minutes. Find the rate of the boat crew in still water and the rate of the current.

Exercise XLIV

180. Solve the following exercises:

1. $\begin{cases} -5s+9p=4 \\ 7s+6p=-80 \end{cases}$

2. $\begin{cases} 25R+14r=385 \\ -15R+9r=30 \end{cases}$

3. $\begin{cases} \frac{2}{3}a-\frac{1}{2}b=10 \\ \frac{7}{5}a+\frac{1}{2}b=32 \end{cases}$

4. $\begin{cases} \frac{m}{6}+\frac{n}{4}=\frac{7}{2} \\ -\frac{m}{8}+\frac{n}{3}=\frac{1}{2} \end{cases}$

5. $\begin{cases} \frac{r-2}{r-3}=\frac{v+4}{v+5} \\ \frac{r+1}{r+2}=\frac{v+3}{v-4} \end{cases}$

6. $\begin{cases} \frac{w-2a+4c}{k-2a+3b}=\frac{3b}{4c} \\ \frac{w+4c}{k+3b}=\frac{2a+3b}{2a+4c} \end{cases}$ (for w and k)

7. $\begin{cases} \frac{ah-ak}{-4a}+\frac{(b-c)h}{4a}-\frac{2(c-b)k}{2a}=b-c \text{ (for } h \text{ and } k) \\ (a-b+c)h=(a+b-c)k \end{cases}$

8. $\begin{cases} \dfrac{25}{a}+\dfrac{6}{b}=6 \\ \dfrac{70}{a}-\dfrac{12}{b}=12 \end{cases}$

9. $\begin{cases} \dfrac{A}{2}+\dfrac{B}{4}-\dfrac{C}{5}=1 \\ \dfrac{A}{3}-\dfrac{B}{2}+\dfrac{C}{3}=12 \\ \dfrac{A}{4}-\dfrac{B}{4}-\dfrac{C}{2}=-13 \end{cases}$

10. $\begin{cases} \dfrac{5}{w}+\dfrac{4}{v}+\dfrac{12}{2r}=4 \\ \dfrac{5}{w}+\dfrac{8}{v}+\dfrac{6}{2v}=4 \\ \dfrac{12}{v}+\dfrac{18}{2v}-\dfrac{10}{w}=4 \end{cases}$

11. $\begin{cases} bC_1+aC_2=4b+a \\ aC_1+bC_2=4a-b \end{cases}$ (for C_1 and C_2)

12. $\begin{cases} (m-n)R_1+(m+n)R_2=2(m^2-n^2) \\ (m+n)R_1+(m-n)R_2=2(m^2+n^2) \end{cases}$ (for R_1 and R_2)

13. $\begin{cases} c-27=\dfrac{17-2c}{3}-\dfrac{5d-8}{2} \\ 16-2d=\dfrac{2c-d}{4}-\dfrac{55+c}{5} \end{cases}$

14. $\begin{cases} \dfrac{2.2g+.8}{1.5}=\dfrac{.5h-.4}{.3} \\ \dfrac{.5g-.3}{.2}=\dfrac{1.5h-.5}{2.5} \end{cases}$

15. $\begin{cases} \dfrac{6-4R}{3}=\dfrac{3H-8}{2} \\ 3R-4=\dfrac{8H-2}{5}-1 \end{cases}$

16. Given $s=\frac{1}{2}at$, and $T=Ra$. (1) Solve for t and a; (2) Solve for R and a.
(3) When in the equations $R=2$, $t=6$, and $s=81$, what do T and a equal?
(4) Check by substituting in the formula obtained by solving (1).

17. Given $l=a+(n-1)d$ and $s=\frac{n}{2}(a+l)$. (1) Solve for a and l; (2) Solve for a and d

Find the values of the remaining letters
when (1), $n=8$, $d=2$, and $s=104$
when (2), $n=16$, $a=5$, and $s=40$
when (3), $a=7$, $l=49$, and $s=812$.

18. Given $l=ar^{n-1}$

and $s=\frac{a(r^n-1)}{r-1}$

(1) Find a and l when $s=765$, $r=2$, and $n=8$
(2) Show that $lr=ar^n$
(3) Show that $r=\frac{s-a}{s-l}$
(4) Find r and n when $a=3$, $l=384$, and $s=765$.

Problems Involving Three Unknown Numbers

181. Some problems lead to systems of three equations in three unknowns.

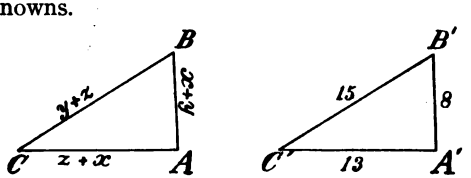


FIG. 234

1. In the triangles A B C and A′B′C′, B = B′, A B = A′B′, and B C = B′C′, and the sides may be designated as shown in Fig. 234. Find the values of x, y, and z.

2. The triangles of Fig. 235 are equal in all respects.

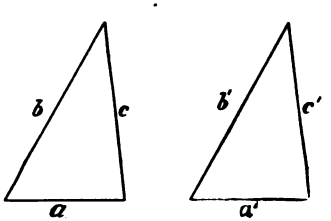


FIG. 235

Find the values of x, y, and z, the sides being designated as follows:

$$(1)\quad \begin{cases} a=x+3y-z, & \text{and } a'=10 \\ b=2x+5y+4z, & \text{and } b'=17 \\ c=3x-y+2z, & \text{and } c'=15 \end{cases}$$

$$(2)\quad \begin{cases} a=2y+3z, & \text{and } a'=14-x \\ b=2(x+z), & \text{and } b'=10-y \\ c=3x+4y, & \text{and } c'=3z+2 \end{cases}$$

$$(3)\quad \begin{cases} a=2(2x+y), & \text{and } a'=13-z \\ b=x+z, & \text{and } b'=y+4 \\ c=x+2y, & \text{and } c'=z \end{cases}$$

$$(4)\quad \begin{cases} a=2x+3z, & \text{and } a'=2(4y-2\tfrac{1}{2}) \\ b=x+5z, & \text{and } b'=4y+8 \\ c=3x-z, & \text{and } c'=5(2y-4). \end{cases}$$

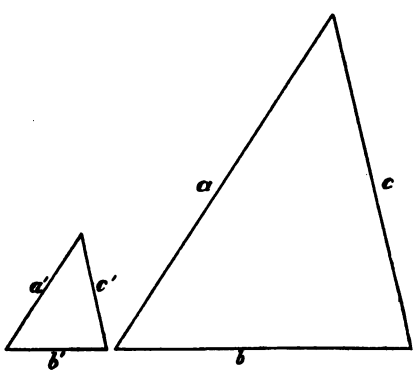


FIG. 236

3. The triangles of Fig. 236 are similar and the ratio of their perimeters is 3. Find the values of x, y, and z under the following conditions:

$$(1)\quad \begin{cases} a=2x+y+2z, & \text{and } a'=7 \\ b=x+4y-z, & \text{and } b'=5 \\ c=-x+3y+5z, & \text{and } c'=6 \end{cases}$$

$$(2)\quad \begin{cases} a=7x+z, & \text{and } a'=10+\dfrac{y}{3} \\ b=x+z, & \text{and } b'=4+\dfrac{y}{3} \\ c=7z+2x, & \text{and } c'=5+\dfrac{y}{3}. \end{cases}$$

4. The sum of the three digits of a number is 16. If the order of the digits is reversed, the new number is 396 less than the original number. If the middle digit be placed first, the resulting number is 90 less than the original number. What is the number?

5. A and B can do a piece of work in 35 days. B and C can do it in $17\frac{1}{2}$ days. C and A can do it in 21 days. How long will it take each to do it alone?

6. In the triangle A B C, Fig. 237, find the values of x, y, and z.

(1) If $a=6$ and $b=8$, $c=8x-7y$
(2) If $a=5$ and $b=12$, $c=5y+3z$
(3) If $a=9$ and $b=12$, $c=4x+3z$.

7. The angles of a triangle are A, B, and C; $\frac{1}{4}A+\frac{1}{8}B=C$, and $\frac{1}{2}A+\frac{1}{10}B=\frac{1}{2}C+30$. Find the values of A, B, and C.

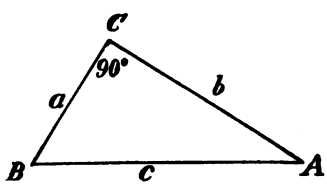


FIG. 237

8. \$18,000 is invested as follows: One part at $3\frac{1}{2}$ per cent., a second part at 5 per cent., and the rest at 4 per cent., and the total annual interest is \$730. If the first part had been invested at 4 per cent., the second at 3 per cent., and the third at 6 per cent.,

the total annual interest would have been $840. How much was each part?

Summary

Problems involving *two unknown numbers* may be solved graphically.

The **graph** of a *linear equation* is a straight line.

The solutions of a **linear equation** are the *coördinates* of points on the **graph** of the equation.

The coördinates of two points are sufficient to locate the graph.

The coördinates of *every point* on the graph of an equation are **solutions** of the equation.

Positive integral solutions and least positive integral solutions of linear equations may be found from the graph of the equation.

The solution of a problem involving two linear equations is shown to be the *coördinates of the point of intersection* of the graphs of the equations.

The graphs of a pair of **dependent equations** are two *parallel* lines.

The conditions of a problem having an **infinite number of solutions** are represented graphically by two coincident lines, the graph of two equivalent equations.

Two equations having all solutions in common are called equivalent equations. Their graphs are coincident lines.

The method of **elimination by addition and subtraction** explained.

The method of **elimination by substitution** explained.

The method of **elimination by comparison** explained.

The solution of literal equations and equations involving the reciprocals of the unknown numbers explained and applied.

A system of two equations, that are linear in the reciprocals

of the unknowns, of the type: $\frac{a}{x}+\frac{b}{y}=\frac{c}{d}$, may be solved as linear equations by regarding $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns.

The solution of systems involving three unknown numbers explained and applied.

Problems involving two and three unknown numbers.

CHAPTER XII

FRACTIONS

182. The operations of addition, subtraction, multiplication, and division are extended to apply to fractions.

Reduction of Fractions

1. How many fourths equal $\frac{1}{2}$? How many sixths? How many eighths?

2. How many halves equal $\frac{6}{12}$? How many fourths? How many sixths?

3. How many sixths equal $\frac{1}{3}$? How many ninths? How many twelfths?

4. How many fourths equal $\frac{18}{24}$? How many eighths? How many twelfths?

5. Show that $\frac{48}{72}=\frac{12}{18}=\frac{2}{3}$.

6. Show that $\frac{a}{ac}$ is equal to $\frac{1}{c}$.

7. Show that $\frac{b}{c}$ is equal to $\frac{ab}{ac}$.

8. Make a rule for reducing fractions to lower terms; to higher terms.

183. *Multiplying numerator and denominator of a fraction by the same number does not alter the value of the fraction.* By this principle we may reduce a fraction to *higher terms.*

184. *Dividing numerator and denominator of a fracion by the same number does not alter the value of the fraction.* By this principle we may reduce a fraction to *lower terms.*

Exercise XLV

185. Reduce the following fractions to lowest terms, doing as many as you can mentally:

1. $\frac{6}{8}$
2. $\frac{12}{27}$
3. $\frac{32}{56}$
4. $\frac{51}{85}$
5. $\frac{91}{126}$
6. $\frac{111}{185}$
7. $\frac{1001}{1443}$
8. $\frac{ab}{bx}$
9. $\frac{abc}{acy}$
10. $\frac{akx^2}{aky^3}$
11. $\frac{a^2b^3m}{a^3b^2m^2}$
12. $\frac{u^3v^5y^6}{u^2v^6y^7}$
13. $\frac{x^n}{x^{n+3}}$
14. $\frac{x^{n+2}}{x^{n+4}}$
15. $\frac{x^{n+3}y^{n+5}}{x^{n+11}y^{n+1}}$
16. $\frac{a(x+y)}{b(x+y)^2}$
17. $\frac{m^3(x-y)^2}{m^5(x-y)}$
18. $\frac{cx+cy}{rx+ry}$
19. $\frac{amn-bmn}{am^3n^2-bm^3n^2}$
20. $\frac{(a+b)^2(b-c)^3}{(c+d)^2(b-c)^2}$
21. $\frac{(u+v)^3(m-n)^7}{(u+v)^5(m-n)^7}$
22. $\frac{a^3b^{11}(r-s)^9}{a^6b^{15}(r-s)^5}$
23. $\frac{a^3b^5c^{79}(a+b)^3(a^2+b^2)^7}{a^2b^2c^{93}(a+b)^7(a^2+b^2)^5}$.

186. *Adding fractions that have the same denominator.*

Exercise XLVI

Add the following fractions mentally:

1. $\frac{2}{7}+\frac{5}{7}$
2. $\frac{1}{5}+\frac{3}{5}$
3. $\frac{1}{3}+\frac{4}{3}-\frac{2}{3}$
4. $\frac{1}{13}+\frac{5}{13}-\frac{3}{13}+\frac{7}{13}$
5. $\frac{7}{9}-\frac{4}{9}-\frac{8}{9}+\frac{6}{9}$
6. $\frac{7}{2}-\frac{5}{2}-\frac{3}{2}-\frac{11}{2}$
7. $\frac{a}{c}+\frac{b}{c}$
8. $\frac{a}{c}-\frac{b}{c}$
9. $\frac{m}{x}+\frac{n}{x}+\frac{k}{x}$
10. $\frac{r}{y}-\frac{s}{y}+\frac{t}{y}-\frac{v}{y}$
11. $\frac{3}{a+b}+\frac{5}{a+b}-\frac{7}{a+b}$.

12. Make a rule for adding fractions having the same denominator.

Exercise XLVII

Combine each of the following expressions into a single fraction:

1. $\frac{12}{23}+\frac{8}{23}+\frac{15}{23}$
2. $\frac{x}{2}-\frac{y}{2}$
3. $\frac{12}{23}+\frac{8}{23}-\frac{15}{23}$
4. $\frac{x}{a}-\frac{y}{a}$
5. $\frac{a}{x}-\frac{b}{x}-\frac{c}{x}$
6. $\frac{a+b}{2}+\frac{a-b}{2}$
7. $\frac{x-y}{a}+\frac{2y}{a}$
8. $\frac{c-ax}{4s}+\frac{c+3ax}{4s}$
9. $\frac{3a}{5b}-\frac{3a-8b}{5b}$
10. $\frac{10x}{17a}+\frac{12x}{17a}-\frac{5x}{17a}$
11. $\frac{5a-b}{x-y}-\frac{2a-3b}{x-y}$.

187. *Adding fractions that have different denominators.*

1. Reduce $\frac{2}{3}$ and $\frac{3}{5}$ to fifteenths.

2. Reduce $\frac{2}{7}$ and $\frac{3}{4}$ to fractions having the same denominator. (See §**183.**)

3. Reduce $\frac{1}{x}$ and $\frac{1}{y}$ to fractions having the same denominator. (See §**183.**)

4. What is the sum of $\frac{1}{2}$ and $\frac{1}{3}$? $\frac{1}{5}$ and $\frac{1}{7}$? $\frac{1}{8}$ and $\frac{1}{9}$? $\frac{1}{10}$ and $\frac{1}{15}$?

5. What is the sum of $\frac{1}{2}$ and $\frac{1}{7}$? $\frac{1}{2}+\frac{1}{7}=\frac{7}{2\cdot 7}+\frac{2}{2\cdot 7}=\frac{7+2}{7\cdot 2}$. Give reasons.

6. In the same way give the sum of $\frac{1}{2}$ and $\frac{1}{3}$. $\frac{1}{5}$ and $\frac{1}{8}$. $\frac{1}{12}$ and $\frac{1}{15}$.

Fractions having 1 for numerator are called *unit fractions.*

7. *Make a rule for writing at once the sum of two unit fractions.*

Exercise XLVIII

188. Using the rule find these sums mentally:

1. $\frac{1}{3}+\frac{1}{4}$
2. $\frac{1}{5}+\frac{1}{6}$
3. $\frac{1}{2}+\frac{1}{5}$
4. $\frac{1}{7}+\frac{1}{9}$
5. $\frac{1}{3}+\frac{1}{x}$
6. $\frac{1}{9}+\frac{1}{b}$
7. $\frac{1}{x}+\frac{1}{y}$
8. $\frac{1}{n}+\frac{1}{m}$
9. $\frac{1}{a+b}+\frac{1}{a}$
10. $\frac{1}{a+b}+\frac{1}{a-b}$
11. $\frac{1}{2a+3b}+\frac{1}{3a-2b}$
12. $\frac{1}{2x^3-5y^2}+\frac{1}{x^3+4y^2}$.

13. What is the difference of $\frac{1}{2}$ and $\frac{1}{3}$? $\frac{1}{4}$ and $\frac{1}{5}$? $\frac{1}{5}$ and $\frac{1}{7}$? $\frac{1}{6}$ and $\frac{1}{9}$? $\frac{1}{2}$ and $\frac{1}{15}$?

14. Write out the work of the parts of problem 13 as is done in this explanation. $\frac{1}{5}-\frac{1}{7}=\frac{7}{5\cdot 7}-\frac{5}{5\cdot 7}=\frac{7-5}{5\cdot 7}$. Give reasons.

15. *Make a rule for writing at once the difference of two unit fractions.*

Exercise XLIX

189. Apply the rule to find these differences:

1. $\frac{1}{3}-\frac{1}{5}$
2. $\frac{1}{4}-\frac{1}{7}$
3. $\frac{1}{9}-\frac{1}{x}$
4. $\frac{1}{3}-\frac{1}{y}$
5. $\frac{1}{b}-\frac{1}{x}$
6. $\frac{1}{c}-\frac{1}{b}$
7. $\frac{1}{t}-\frac{1}{m}$

8. $\frac{1}{b}-\frac{1}{a+b}$

9. $\frac{1}{a+b}-\frac{1}{a-b}$

10. $\frac{1}{2a+3b}-\frac{1}{3a+2b}$

11. $\frac{1}{x^2+y^2}-\frac{1}{x^2-y^2}$

12. $\frac{1}{a+b}-\frac{1}{a+b+c}$

13. $\frac{2}{3}-\frac{5}{a+b}$.

14. Show that the addition and subtraction of unit fractions may be indicated thus: $\frac{1}{5}\pm\frac{1}{11}=\frac{11\pm 5}{5\cdot 11}$.

190. Addition and Subtraction of Fractions

1. Add $\frac{5}{6}$ and $\frac{7}{8}$. $\frac{5}{6}+\frac{7}{8}=\frac{5\cdot 8}{6\cdot 8}+\frac{6\cdot 7}{6\cdot 8}=\frac{5\cdot 8+6\cdot 7}{6\cdot 8}$.

2. Subtract $\frac{4}{7}$ from $\frac{8}{9}$.

3. Add $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ where n_1 (read "n one") and d_1 (read "d one") stand for first numerator and first denominator; n_2 (n two) and d_2 (d two) stand for second numerator and second denominator.

4. $\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_1d_2}{d_1d_2}+\frac{n_2d_1}{d_1d_2}=\frac{n_1d_2+n_2d_1}{d_1d_2}$; give reasons.

5. Subtract $\frac{n_2}{d_2}$ from $\frac{n_1}{d_1}$.

6. From 3 and 4 obtain a rule for writing at once the sum or difference of two fractions.

7. By this rule write the sum and difference of $\frac{a}{b}$ and $\frac{c}{d}$.

Exercise L

By this rule give the values of the following indicated sums and differences, giving as many as you can mentally:

1. $\frac{5}{8}+\frac{4}{11}$
2. $\frac{6}{7}-\frac{3}{4}$
3. $\frac{5}{12}+\frac{2}{11}$
4. $\frac{x}{3}\pm\frac{7}{8}$
5. $\frac{a}{9}\pm\frac{7}{12}$
6. $\frac{a}{5}\pm\frac{c}{10}$
7. $\frac{x}{a}\pm\frac{y}{a}$
8. $\frac{8}{x}\pm\frac{7}{y}$
9. $\frac{x}{y}\pm\frac{c}{d}$
10. $\frac{x}{a}\pm\frac{x}{b}$
11. $\frac{a+3}{5}+\frac{a+5}{7}$
12. $\frac{a+x}{a-x}-\frac{a-x}{a+x}$
13. $\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$
14. $\frac{x^2+2a^2}{x^2+a^2}-\frac{x^2-2a^2}{x^2-a^2}$.

15. Add $\frac{5}{12}+\frac{3}{8}$.

$$\frac{5}{12}+\frac{3}{8}=\frac{5\cdot 8+3\cdot 12}{8\cdot 12}=\frac{76}{8\cdot 12}=\frac{19}{24}.$$

This method is especially useful in adding two fractions whose denominators have no common factors.

In case the denominators of the fractions to be added have a common factor, the fractions should be reduced to fractions having for their denominators the least common multiple of the given denominators, and then added, thus:

$$\frac{5}{12}+\frac{3}{8}=\frac{10}{24}+\frac{9}{24}=\frac{19}{24}.$$

16. This method is used in adding three or more fractions, thus:

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{10}=\frac{1\cdot 5}{2\cdot 5}+\frac{1\cdot 2}{2\cdot 5}+\frac{1}{10}=\frac{8}{10}=\frac{4}{5}.$$

17. Add $\frac{a}{x^2}+\frac{b}{xy}-\frac{c}{y^2}$.

$$\frac{a}{x^2}+\frac{b}{xy}-\frac{c}{y^2}=\frac{ay^2}{x^2y^2}+\frac{bxy}{x^2y^2}-\frac{cx^2}{x^2y^2}=\frac{ay^2+bxy-cx^2}{x^2y^2}.$$

Exercise LI

191. Simplify the following:

1. $\frac{p}{2}+\frac{q}{5}+\frac{r}{10}$
2. $\frac{4b}{3}+\frac{5c}{6}+\frac{7d}{2}$

3. $\frac{5a}{12}-\frac{3a}{20}$

4. $\frac{1}{a}-\frac{y}{ax}$

5. $\frac{a}{cx}+\frac{b}{cy}$

6. $\frac{1}{c}-\frac{b}{ca}$

7. $\frac{x}{12ab}-\frac{z}{6b}$

8. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$

9. $\frac{2}{xy}-\frac{3y^2}{xy^3}+\frac{xy^5+y^3}{x^2y^6}$

10. $\frac{a}{a-1}-\frac{ab}{a(a-1)}$

11. $\frac{1}{x-1}-\frac{1}{2(x-1)}$

12. $\frac{1}{2x-3y}+\frac{x+y}{4x^2-6xy}$

NOTE.—
$4x^2-6xy=2x(2x-3y)$

13. $\frac{b}{a+2b}+\frac{ab}{3ad+6bd}$

14. $\frac{a}{x}+b$

NOTE.—Put $b=\frac{b}{1}$.

15. $\frac{x}{y}-a$

16. $5a+\frac{2a}{3}$

17. $11x-\frac{7x}{3}$

18. $\frac{3x}{4}-x$

19. $\frac{a}{x^2y}+\frac{b}{xy^2}$

20. $\frac{a+b}{x}+\frac{a-b}{3x}$

21. $\frac{a-2b}{3x}-\frac{4a-5b}{5x}$

22. $\frac{1}{a^3}-\frac{1}{a^2}+\frac{1}{a}$

23. $\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}$

24. $\frac{1}{x-y}-\frac{1}{y}$

25. $\frac{5x-4y+3z}{x}+\frac{2x+3y-4z}{3y}$

26. $\frac{7a+3b-4c}{a}-\frac{2b+4a-3c}{b}$

27. $\frac{x}{x^2y}-\frac{4}{x^2y^2}+\frac{z}{xy^2}$

28. $\frac{a+b}{a}-\frac{a}{a-b}$

29. $\frac{(a+b)^2}{4ab}-1$

30. $\frac{x}{x^2-1}+\frac{x+3}{x-1}-\frac{x-3}{x+1}$

31. $\frac{3a}{c+d}=\frac{a}{c-d}-\frac{2ac}{c^2-d^2}$.

192. Multiplication of Fractions

1. Multiply $\frac{3}{4}$ by 8; by 12; by 5; by 25; by a; by xy.

2. Multiply $\frac{1}{4}$ by $\frac{1}{2}$; $\frac{1}{3}$ by $\frac{1}{5}$; $\frac{2}{3}$ by $\frac{1}{5}$; $\frac{1}{a}$ by $\frac{1}{3}$; $\frac{1}{a}$ by $\frac{2}{3}$; $\frac{1}{a}$ by $\frac{1}{b}$ ($\frac{1}{2}$ of $\frac{1}{4}$, $\frac{1}{2}\times\frac{1}{4}$, and $\frac{1}{2}\cdot\frac{1}{4}$ are all equivalent).

3. Multiply $\frac{3}{4}$ by $\frac{5}{7}$; by $\frac{4}{5}$; by $\frac{3}{5}$; by $\frac{4}{3}$; by $\frac{a}{5}$; by $\frac{a}{3}$; by $\frac{a}{b}$.

4. Make a rule for multiplying two fractions together.

Fractions are multiplied by multiplying their numerators for the numerator of the product, and multiplying their denominators for the denominator of the product.

5. Multiply $\frac{12}{35}\cdot\frac{15}{16}$.

Since $12\cdot 15$ will be the numerator of the product (why?), and $35\cdot 16$ will be the denominator of the product (why?), any factor of 12 or 15 which is also a factor of 35 or 16 will divide both the numerator and the denominator of the product. It is simpler to divide out, or *cancel*, such factors in advance, thus:

$$\frac{\overset{3}{\cancel{12}}}{\underset{7}{\cancel{35}}}\cdot\frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{16}}}=\frac{3}{7}\cdot\frac{3}{4}=\frac{9}{28}.$$ Give reasons.

Exercise LII

193. Simplify the following:

1. $\frac{2}{3}\times\frac{4}{5}$
2. $\frac{1}{4}\times\frac{2}{3}$
3. $\frac{4}{5}\times\frac{8}{9}$
4. $\frac{5}{6}\times\frac{7}{8}$
5. $\frac{7}{8}\times\frac{2}{5}$
6. $\frac{a}{5}\times\frac{b}{7}$
7. $\frac{a}{b}\times\frac{c}{d}$
8. $\frac{x}{y}\times\frac{c}{d}$
9. $\frac{s}{t}\times\frac{p}{q}$
10. $\frac{a}{x}\times\frac{c}{a}$
11. $\frac{n_1}{d_1}\times\frac{n_2}{d_2}$
12. $\frac{a_1}{b_1}\times\frac{a_2}{b_2}$.

Exercise LIII

194. Find the values of these products, using cancellation when possible:

1. $\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7}$
2. $\frac{3}{4} \cdot \frac{5}{7} \cdot \frac{7}{8}$
3. $\frac{3}{4} \cdot \frac{5}{7} \cdot \frac{a}{b}$
4. $\frac{5}{6} \cdot \frac{x}{y} \cdot \frac{r}{t}$
5. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{x}{y}$
6. $\frac{n_1}{d_1} \cdot \frac{n_2}{d_2} \cdot \frac{n_3}{d_3}$
7. $\frac{a}{b^2} \cdot \frac{a^2}{b} \cdot \frac{a^3}{b^5}$
8. $\frac{a^2}{b^3} \cdot \frac{b^2}{a^3} \cdot \frac{b}{a}$
9. $\frac{3}{a^2} \cdot \frac{a^3}{6} \cdot \frac{4}{a}$
10. $\frac{a}{x} \cdot \frac{x^3}{a^2} \cdot \frac{a^2}{x^2}$.

9. Make a rule for multiplying any three fractions together.

10. Change your rule to make it apply to the product of 4 fractions; of n fractions.

11. Make a rule for dividing out or cancelling factors in the multiplication of any number of fractions.

Exercise LIV

195. Perform the indicated operations, using cancellation when possible:

1. $\frac{2}{5} \cdot \frac{3}{5} \cdot \frac{1}{8}$
2. $\frac{68}{102} \cdot \frac{95}{133}$
3. $\frac{1}{a} \cdot ac$
4. $\frac{1}{a^2} \cdot a^3$
5. $\frac{a}{b} \cdot \frac{1}{x}$
6. $\frac{a}{c} \cdot \frac{c}{d}$
7. $\frac{7xyz}{3bc} \cdot 9abc$
8. $\frac{ab}{xy} \cdot \frac{yz}{bc}$
9. $\frac{2ab}{3xy} \cdot \frac{5ax}{6by}$
10. $\frac{15ab}{16xy} \cdot \frac{24xyz}{25bc}$
11. $\frac{3ab}{4xy} \cdot \frac{5bc}{6yz} \cdot \frac{7xz}{8ac}$
12. $\frac{2a^2x}{3b^2y} \cdot \frac{6by^2}{7ax^2} \cdot \frac{5b}{4a}$
13. $\frac{a^4b^4}{4x^2y^3} \cdot \frac{6x^4y^2}{5a^4b^2}$
14. $\frac{a+b}{a-b} \cdot \frac{a^2-b^2}{a^2+b^2}$
15. $\frac{27x}{8y+8x} \cdot \frac{x+y}{3}$

16. $\frac{a}{a+b} \cdot \frac{b}{a-b}$

17. $\frac{6(x-y)}{5xy^2} \cdot \frac{15x^2y^3}{8(x-y)}$

18. $\frac{a^2-ab}{x^2-xy} \cdot \frac{x^2+xy}{a^2+ab}$

19. $\frac{a+1}{a-1} \cdot \frac{a+2}{a-2} \cdot \frac{a+3}{a-3}$

20. $\frac{x^2-xy}{3a+3b} \cdot \frac{(a+b)^2}{(x-y)^2}$.

21. Show that $-\frac{a}{b} = +\frac{-a}{b} = +\frac{a}{-b}$.

Division of Fractions

Exercise LV

196. Multiply the following fractions:

1. $\frac{2}{3} \cdot \frac{3}{2}$
2. $\frac{3}{8} \cdot \frac{5}{3}$
3. $\frac{7}{9} \cdot \frac{9}{7}$
4. $\frac{x}{3} \cdot \frac{3}{x}$
5. $\frac{x}{10} \cdot \frac{10}{x}$
6. $\frac{x}{y} \cdot \frac{y}{x}$
7. $\frac{n}{d} \cdot \frac{d}{n}$
8. $\frac{4}{1} \cdot \frac{1}{4}$
9. $\frac{12}{1} \cdot \frac{1}{12}$
10. $\frac{a}{1} \cdot \frac{1}{a}$
11. $\frac{x^5}{y} \cdot \frac{y}{x^5}$
12. $\frac{a+b}{c+d} \cdot \frac{c+d}{a+b}$.

13. What is the simplest form of the product in each of the problems 1–12?

14. By examining the products of problems 1–12, tell without actually dividing, how many times $\frac{2}{3}$ is contained in 1; $\frac{3}{2}$ in 1; $\frac{3}{8}$ in 1; $\frac{5}{3}$ in 1; $\frac{4}{7}$ in 1; $\frac{7}{4}$ in 1; 2 in 1; 12 in 1; a in 1; $\frac{b}{a}$ in 1.

15. Give a quick way of finding how many times any fraction, or any whole number is contained in 1.

16. How many times are these numbers contained in 1?

(1) $\frac{4}{9}$ (2) $\frac{6}{7}$ (3) $\frac{7}{8}$

(4) $\frac{6}{11}$ (6) $\frac{3}{4}$ (8) $\frac{x}{y}$

(5) $\frac{x}{3}$ (7) $\frac{a}{b}$ (9) $\frac{a}{x}$.

17. A certain number is contained in 1, $\frac{3}{5}$ times; how many times is it contained in 2? in 3? in 5? in 8? in 20? in a? in b? in $\frac{3}{4}$? in $\frac{5}{9}$? in $\frac{9}{16}$? in $\frac{a}{b}$? in $\frac{c}{d}$? in $\frac{x}{y}$?

18. How many times is $\frac{5}{7}$ contained in the following numbers?

(1) in $\frac{3}{8}$? (4) in $\frac{9}{10}$? (7) in a?

(2) in $\frac{5}{7}$? (5) in $\frac{7}{15}$? (8) in $\frac{a}{b}$?

(3) in $\frac{7}{8}$? (6) in 12? (9) in $\frac{x}{y}$?

19. Find the result in the following indicated divisions:

(1) $\frac{3}{5} \div \frac{2}{7}$ (3) $\frac{7}{9} \div \frac{6}{7}$ (5) $\frac{a}{b} \div \frac{c}{x}$

(2) $\frac{4}{7} \div \frac{6}{11}$ (4) $\frac{3}{8} \div \frac{7}{5}$ (6) $\frac{c}{x} \div \frac{d}{f}$.

One fraction is divided by another by multiplying the dividend by the inverted divisor; that is, by multiplying the dividend by the reciprocal of the divisor.

Why is the divisor inverted?

Why is the inverted divisor multiplied by the dividend?

Exercise LVI

197. Perform the indicated operations:

1. $x \div \frac{y}{x}$
2. $ab \div \frac{xy}{z^2}$
3. $12x^3 \div \frac{16x^2}{7y}$
4. $25a^6 \div \frac{10a^2}{9x^2}$

5. $18x^2y^2z^4 \div \frac{15x^3y^3z}{ab}$

6. $(-ab) \div \frac{3d}{5ab}$

7. $(32x^3y^2z^2 - 40x^2y^3z^2) \div \frac{8x^3y^3z^3}{a^3b^2}$

8. $\frac{16a^3b^2}{27x^5y^3} \div \frac{8a^3b}{9x^2y^4}$

9. $\frac{20x^2y^3}{21a^4c^5} \div \frac{4axy}{3a^2b^2c^2}$

10. $\frac{40a^7b^5c^6}{22m^3x^4z^5} \div \frac{35a^6b^6c^6}{88m^6xz^7}$

11. $\frac{x^r}{x^{n+1}} \div \frac{x^{r-2}}{x^{n+3}}$

12. $\frac{a^{k-2}}{b^{m+1}} \div \frac{a^{k+3}}{b^{m-2}}$

13. $\frac{24(x-1)^2}{70(a^2-b^2)} \div \frac{30(x-1)}{28(a-b)^2(a+b)}$

14. $\frac{14x^2-7x}{12x^3+24x^2} \div \frac{2x-1}{x^2+2x}$.

Summary

Multiplying both the numerator and the denominator of a fraction by the same number *does not alter the value* of the fraction.

Dividing both the numerator and the denominator of a fraction by the same number *does not alter the value* of the fraction.

Fractions having the same denominator are **added** or **subtracted** by dividing the sum or the difference of the numerators by the common denominator.

Fractions having different denominators are **added** or **subtracted** by first reducing them to equivalent fractions having a common denominator (usually the least common denominator) and then adding or subtracting the resulting fractions.

Fractions are **multiplied** by multiplying their numerators for the numerator of the product, and their denominators for the denominator of the product.

Fractions are **divided** by multiplying the dividend by the inverted divisor.

Cancelling, or removing, the same factor from both the numerator and the denominator of a fraction is equivalent to **dividing** both the numerator and the denominator by the same number.

The numerator and the denominator, when spoken of together, are called the **terms** of the fraction.

CHAPTER XIII

FACTORING. QUADRATICS. RADICALS

198. The factors of a number are its exact divisors. The process of factoring is therefore inverse to the process of multiplication. In this chapter we consider only factors which are non-fractional (integral) and which do not involve radical signs (i. e., which are rational numbers). We, therefore, admit only those factors of a number which divide it exactly with a *rational integral* quotient.

A **prime number** is a number which has no factors except *itself* and *unity*. Give examples.

A **prime factor** is a *factor that is a prime number.*

A number has only one set of prime factors.

Monomial Factors

1. $a(b+c)=ab+ac$, therefore the factors of $ab+ac$ are a and $b+c$. The monomial a appears as a *common factor* of the terms of $ab+ac$.

2. $x^2y(k+l+m)=kx^2y+lx^2y+mx^2y$, therefore the factors of $kx^2y+lx^2y+mx^2y$ are x^2y and $k+l+m$. The monomial x^2y appears as a common factor of the terms of the given expression $kx^2y+lx^2y+mx^2y$.

3. Multiply $3x^2y^3$ and $(2xy-3x^3-5y^2)$. What are the factors of $6x^3y^4-9x^5y^3-15x^2y^5$? How does the monomial $3x^2y^3$ appear in the expression $6x^3y^4-9x^5y^3-15x^2y^5$?

4. Factor $3a^2b-12ab^2$.

Since $3ab$ is a factor of each term, it is a factor of the whole expression (each term in succession). We obtain the quotient $a-4b$. Then $3ab$ and $a-4b$ are the factors of $3a^2b-12ab^2$, or $3a^2b-12ab^2=3ab(a-4b)$.

Test by multiplication:

$$\begin{array}{r} a-4b \\ 3ab \\ \hline 3a^2b-12ab^2 \end{array}.$$

199. *An expression is equal to the product of all of its prime factors for all values of its letters.* It will be recalled that such equality is called *identity.* (See p. 213.)

By this principle, we can test, or verify the correctness of the factors. If the factors of $3a^2b-12ab^2$ are $3ab$ and $a-4b$, then

$$3a^2b-12ab^2=3ab(a-4b),$$

for *all* values of a and b. The two expressions are therefore equal for the values, $a=5$ and $b=1$.

Test by substitution:

For $a=5$ and $b=1$;

$$3a^2b-12ab^2=3\times5^2\times1-12\times5\times1^2=15$$

$$\text{and } 3ab(a-4b)=3\times5\times1(5-4)=15.$$

Both $3a^2b-12ab^2$ and $3ab(a-4b)$ being 15 for $a=5$ and $b=1$, the expression is correctly factored.

Verify by substituting other values for a and b.

Exercise LVII

200. Factor the following expressions and test results both by multiplication and by substitution, doing as many as you can mentally:

1. $3x+3y$
2. am^2+bm^2
3. $abc-abd$
4. $3nx+6ny$
5. $2cdl+4cdm$
6. ax^2y+bx^2y
7. $cx^2+dx^3+fx^4$
8. $3xyz+12x^2yz+9xy^2z^2$
9. $5a^3b+25a^4c-10a^5d$
10. $12a^2b^2r+18a^3b^4s+6a^4b^5t$
11. $8x^3y^2+4x^2y^3$
12. $3x^2y^2-2xy-3xy^3$

13. $15a^3x-10a^3y+5a^3z$
14. $32a^3b^3-ab^6$
15. $6x^3y^2+3x^2y^3+3xy^4$
16. $8x^2y^2+16xyz-24x^2y^2z^2$
17. $2a^2b^2+a^3b^3+ab^4$
18. $8a^3b^2c^2-4a^2b^3c^3+a^2b^2c^3$
19. $15x^2+20xy+5y^2$
20. $16a^2b^2+48a^2b-16ab+8a$
21. $21n^2s+35ns^2+56ns$
22. $39u^5v^7+26u^8v^{10}-91u^{11}v^9$
23. $48f^2g^2h^3-60f^3g^3h+54f^2gh^2$
24. $6ax^2y-8ax^3-5ay^2-4axy$
25. $77m^4n^4-22m^3n^5+33m^5n^3-55m^6n^7$
26. x^n+x^{2n}
27. x^a+x^{a+b}
28. $a^xb^{2y}+a^{2x}b^y$
29. $15a^{5x}b^{2y}+25a^{4x}b^{3y}$
30. $x^{2m}y^{3n}-3x^{3m}y^{2n}+2x^{2m}y^n$
31. $k^{a+b}l^{b+c}-k^{b+c}l^{a+c}$
32. $6m^{2r+3s}n^{3r+5s}-9m^{2r+s}n^{r+5s}$.

Exercise LVIII

201. Reduce to lowest terms:

1. $\dfrac{3b+3c}{5b+5c}$
2. $\dfrac{ab+ac}{bx+cx}$
3. $\dfrac{6x+6y}{12a+12b}$
4. $\dfrac{5x^2-10xy}{4ax-8ay}$
5. $\dfrac{8x^3y^2+4x^2y^3}{14x^2yz+7xy^2z}$
6. $\dfrac{5a^2b-10ab^2}{15abc+20a^2b^2}$
7. $\dfrac{2km^2-6bm^2}{3kn^3-9bn^3}$
8. $\dfrac{5ad^2-15bd^2-10cd^2}{6axy^2-18bxy^2-12cxy^2}$.

Perform the operations indicated in the following problems:

9. $\dfrac{3x^2}{5x-10} \cdot \dfrac{2x-4}{6x^4}$

10. $\dfrac{ab+ac}{bd-cd} \cdot \dfrac{ab-ac}{bd+cd}$

11. $\dfrac{3m-3n}{5m+5n} \div \dfrac{9m-9n}{10m+10n}$

12. $\dfrac{5}{x-3}+\dfrac{7}{3x-9}$

13. $\dfrac{1}{ax+ay}+\dfrac{2}{bx+by}-\dfrac{3}{cx+cy}$

14. $\dfrac{2}{4x-8y}-\dfrac{3}{x-2y}+\dfrac{x+10-2y}{4x-8y}$.

Exercise LIX

202. Solve the following equations:

1. Solve $mx+nx=am+an$ for x.

   Factoring, $x(m+n)=a(m+n)$.
   Then $x=a$. Why? Check.

2. $ax+bx=ac+bc$. Solve for x
3. $5a^2y-3b^2y=10a^2m-6b^2m$. Solve for y
4. $kv+lv-3mv=5kr^2s+51r^2s-15mr^2s$. Solve for v
5. $ak+bm=am+bk$. Solve for k
6. $abx-abd=cx-cd$. Solve for x
7. $mt-6tw^3=5ms-30sw^3$. Solve for m
8. $ax+bx=cx+d$. Solve for x
9. $a+b=\dfrac{ac}{y}+\dfrac{bc}{y}$. Solve for y
10. $m^2+\dfrac{abn^2}{k}=n^2+\dfrac{abm^2}{k}$. Solve for k
11. $u+v=\dfrac{a}{x}$. Solve for x.

203. Solve the following problems:

1. In 5 quarts of a certain solution there is m times as much water as chemicals. How much water and how much chemicals in the solution?

2. In an alloy of two metals, the ratio of the weight of one metal to the weight of the other is $7:k$. There are 3 lb.

more of the first metal than of the second in the alloy. Find the weight of each in the alloy.

3. Two numbers have the ratio $a:b$, and their sum is c. Find the numbers.

4. Two numbers have the ratio $m:n$, and their difference is equal to the sum of u and v. Find the numbers.

5. The second of three numbers is r times the first, and the third is s times the second. Find the three numbers, if their sum is equal to the sum of the numbers a, ar, and ars.

204. Polynomials whose terms may be grouped to show a common binomial factor.

1. Multiply $a+b$ by $m+n$.

$$\begin{array}{r} a+b \\ m+n \\ \hline am+bm+an+bn. \end{array}$$

In this multiplication $a+b$ is first multiplied by m, then by n, and the products are added. Therefore,

$$(a+b)\ (m+n)=m(a+b)+n(a+b)=am+bm+an+bn.$$

2. Factor $3a+3b+5a+5b$.

$$3a+3b+5a+5b=3(a+b)+5(a+b)=(a+b)(3+5).$$

This method of factoring is called *grouping*.

3. Factor $ac+bc+ad+bd$.

$$ac+bc+ad+bd=c(a+b)+d(a+b)=(a+b)(c+d).$$

Test by multiplication:

$$\begin{array}{r} c+d \\ a+b \\ \hline ac+ad+bc+bd \end{array}.$$

Test by substitution:

Let $a=1$; $b=2$; $c=3$; $d=4$.

Then $ac+bc+ad+bd=1\cdot3+2\cdot3+1\cdot4+2\cdot4=$
$3+6+4+8=21$

and, $(c+d)(a+b)=(3+4)(1+2)=7\times3=21$.

Therefore, $ac+bc+ad+bd=(c+d)(a+b)$.

4. Factor $14x^3-6x^2-21x+9$.

$14x^3-6x^2-21x+9=2x^2(7x-3)-3(7x-3)=(7x-3)(2x^2-3)$.

Test by substitution and by multiplication.

Exercise LX

205. Resolve into factors the following expressions and test results, doing as many as you can mentally:

1. $ax+bx+am+bm$
2. $ar+br+as+bs$
3. $ad+bd+at+bt$
4. $3a+3b+ay+by$
5. $ak-bk+al-bl$
6. $ax^2-bx^2+ay^3-by^3$
7. $abc+abx+nc+nx$
8. $a^2k+a^2l+b^2k+b^2l$
9. $5au-5av+mu-mv$
10. $m^2a+ma^2+m^3a^2+m^2a^3$
11. $a^2-ad+ab-bd$
12. $x^6+5x^4+x^3+5x$
13. $6x^2-9x-10xy+15y$
14. $2m^3+m^2+6m+3$
15. $3ac+3ax-5c-5x$
16. $9-15r+27r^2-45r^3$
17. $8gh+12ah+10bg+15ab$
18. $15z-6-20zw+8w$
19. $2m^2+3km-14mn-21kn$
20. $3ax+3ab+2x^2+2bx+b+x$
21. $4x^3+4x-4x^2z-4z$
22. $1+r-r^2xy-r^3xy$
23. x^2-x^3+1-x
24. $(a+m)(c+n)-2n(a+m)$
25. $(x+y)(a+b)-(x+y)(b+c)$
26. $m(x+y)^2+(x+y)$
27. $a^{x+m}+a^mb^y+a^xb^m+b^{m+y}$
28. $2x^{r+1}-3xy^s-2x^ry+3y^{s+1}$
29. $3m^{u+2}-5m^un^3+3m^2n^v-5n^{v+3}$
30. $k^{n+5}-k^{n+3}-k^{n+1}+k^{n-1}$.

Exercise LXI

206. Reduce to lowest terms:

1. $\dfrac{ax+bx+am+bm}{ar+br+as+bs}$
2. $\dfrac{3u-3v+au-av}{5bu-5bv+2ku-2kv}$
3. $\dfrac{ax^2-bx^2+ay^2-by^2}{mx^2+my^2+nx^2+ny^2}$
4. $\dfrac{x^4-2x^3+7x-14}{2x^3-4x^2+6x-12}$
5. $\dfrac{2am-3at-10bm+15bt}{10am-15at-2bm+3bt}$.

Perform the indicated operations in the following:

6. $\dfrac{3a-3b}{5c-5d} \cdot \dfrac{cx-dx+cy-dy}{am-bm+an-bn}$

7. $\dfrac{x^2-xy-6x+6y}{ax-bx-ay+by} \cdot \dfrac{5ac-5bc}{xy-6y}$

8. $\dfrac{1}{ab+ac-bk-ck} + \dfrac{1}{bm+cm+bk+ck}$

9. $\dfrac{2am-2bm-5an+5bn}{2cm-5cn+2dm-5dn} \div \dfrac{5ax-5bx}{6cy+6dy}$.

Exercise LXII

207. Solve the following equations:

1. $ax+bx=ad+bd+ac+bc$. Solve for x
2. $3a^2bx+12bcd=4ab^2x+9acd$. Solve for x
3. $ax-bc-ac=ab+bx$. Solve for x
4. $am+bm=3am+3bm+ak+bk-an-bn$. Solve for m
5. $av+bv-3cv=5a+5b-15c+2aw+2bw-6cw$. Solve for v. Solve for w
6. $a-b=\dfrac{ad}{y}+\dfrac{ac}{y}-\dfrac{bc}{y}$. Solve for y
7. $1+\dfrac{b}{a}-\dfrac{d}{x}=\dfrac{bd}{ax}+\dfrac{c}{x}+\dfrac{bc}{ax}$. Solve for x
8. $\dfrac{bm}{6cd}+\dfrac{2}{5b}=\dfrac{am}{2cd}+\dfrac{2}{15a}$. Solve for m
9. $\dfrac{10}{2u+1}-\dfrac{3}{4u+2}=1$. Solve for u
10. $\dfrac{4}{x^3+x^2+x+1}-\dfrac{2}{x^3-3x^2+x-3}=0$. Solve for x
11. $\dfrac{1}{2y-2}+\dfrac{2}{3y-3}+\dfrac{3}{4y-4}=4$. Solve for y.

208. Solve the following problems:

1. In one pile of brick there are r times as many bricks as in a second pile. If placed in three piles of a, r, and ar

bricks respectively, there is one brick left over. How many bricks in each of the two original piles?

2. In an alloy of two metals, the ratio of the weights of the metals is $a:b$. A certain mass of the alloy is $a+b$ pounds more than another mass, and the weight of the two masses together is $3a+3b$. How many pounds of each metal are there in each of the masses of alloy?

The Trinomial Square

EXERCISE LXIII

209. Multiply:

1. $(a+b)^2$	8. $(m-n)^2$	15. $(3m^3-5n)^2$
2. $(r+s)^2$	9. $(x+5)^2$	16. $(ab+3x^2)^2$
3. $(c+d)^2$	10. $(y-3)^2$	17. $(a^2m-5x^3)^2$
4. $(a-b)^2$	11. $(7-a)^2$	18. $(abc^3+3x^2y)^2$
5. $(g-h)^2$	12. $(2u-3v)^2$	19. $(x^2y-7xy^3)^2$
6. $(r-s)^2$	13. $(4b+7y)^2$	20. $(3r^2s^3-4rs^2)^2$
7. $(c-d)^2$	14. $(3x^2+2y)^2$	21. $(x+\sqrt{2})^2$.

22. From these exercises make a rule for squaring a binomial.

When a binomial is multiplied by itself the result is a trinomial which consists of the square of the first term of the binomial, plus or minus (according as the binomial is a sum or a difference) twice the product of the first and second terms of the binomial, plus the square of the second term of the binominal.

Then any trinomial, in which two terms are squares (and positive) and the other term is plus or minus twice the product of the square roots of those two terms, is the square of the sum or difference of those two square roots according as the third term is plus or minus.

23. Factor $k^2+6kl+9l^2$.

k^2 and $9l^2$ are squares, and $6kl$ is twice the product of their square roots, therefore, $k^2+6kl+9l^2=(k+3l)^2$.

Exercise LXIV

210. Factor the following expressions, and test each result, doing as many as you can mentally:

1. $x^2+2xy+y^2$
2. $a^2-2ab+b^2$
3. $m^2+4mn+4n^2$
4. $4a^2-4a+1$
5. b^2-6b+9
6. $k^2+8k+16$
7. $a^2+10ab+25b^2$
8. $4m^2-12am+9a^2$
9. $25+80r+64r^2$
10. $121a^2+198ay+81y^2$
11. $c^2-16c+64$
12. x^4+30x^2+225
13. $49-140n^2+100n^4$
14. $a^2b^2c^2+8abc+16$
15. $49x^6-154x^3y+121y^2$
16. $625m^4+50m^2v^2+v^4$
17. $16k^2+72kvw+81y^2w^2$
18. $169r^2-78rt^2+9t^4$
19. $225s^{12}-420s^6c^5+196c^{10}$
20. $(m+n)^2+2(m+n)+1$
21. $(u+v)^2+4t(u+v)+4t^2$
22. $9k^2+6k(r+s)+(r+s)^2$
23. $(a+b)^2+2(a+b)(c+d)+(c+d)^2$
24. $m^2+2mn+n^2+2m+2n+1$
25. $m^2+2mn+n^2+6am+6an+9a^2$
26. $a^2+b^2+c^2+2ab+2ac+2bc.$

27. Square 53.

$$53^2=(50+3)^2=50^2+2\cdot 50\cdot 3+3^2=2{,}500+300+9=2{,}809.$$

28. Square 68.

$$68^2=(70-2)^2=4{,}900-280+4=4{,}624.$$

29. Square the following numbers mentally:

(1) 13	(5) 22	(9) 43	(13) 91
(2) 14	(6) 31	(10) 38	(14) 89
(3) 15	(7) 19	(11) 72	(15) 67
(4) 21	(8) 18	(12) 81	(16) 103.

Exercise LXV

211. Factor the following by any method, and test:

1. $a^2k^3-2abk^3+b^2k^3$
2. $x^4y+x^2y^3-x^3y^2-xy^4$
3. ax^3+ax^2+ax+a
4. $7bx^2+42bxy+63by^2$

5. $12a^2xy+12axy+3xy$
6. $5x^5+5x^4+4x^3+4x^2+3x+3$
7. $(c+d)(c^2+d^2)+2c^2d+2cd^2$
8. $x^2-6xy+9y^2-2xz+6yz+z^2$
9. $9a^2m+9a^2n+12am+12an+4m+4n$
10. $(x+y)^2(x-y)-(x-y)^2(x+y)$.

Quadratic Expressions and Quadratic Equations

212. An **expression** in which the highest power of the unknown is the second power is called a *second-degree expression*, or a *quadratic expression*.

213. An **equation** in which the highest power of the unknown is the second power, is called a *second-degree equation* or a *quadratic equation*.

For example, $x^2-10x+21$ and y^2-25 are second degree, or quadratic, expressions; while $x^2-10x+21=0$, and $y^2-25=0$ are second degree, or quadratic, equations.

214. Graph, on cross-lined paper, the quadratic expression, $x^2-10x+21$.

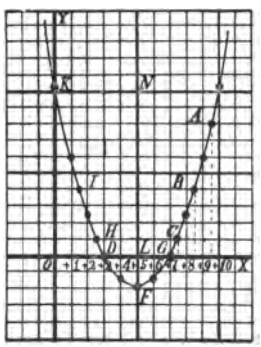


FIG. 238

x	$x^2-10x+21$
0	+21
+1	+12
+2	+5
+3	0
+4	−3
+5	−4
+6	−3
+7	0
+8	+5
+9	+12
+10	+21

Plot the points of the table and draw through them a smooth curve. The scale of Fig. 238 is horizontally, 1 space = 1, and

vertically, 1 space $=2$. What is the value of $x^2-10x+21$ at the point A? At B? At C? At G? At D? At H?

What is the value of x at the point A? At B? At C? At G? At D? At H? At I?

The vertical distance of any point from the horizontal axis represents the value of the expression, $x^2-10x+21$, at that point.

The horizontal distance from o to the foot of a perpendicular to the horizontal through any point of the curve represents the value of x for that point.

The lines O X and O Y of Fig. 238 are called **axes.**

215. Graph the following quadratic expressions between the indicated limits for x and keep the graphs for reference:

1. $x^2-8x+12$, from o to $+8$
2. x^2-6x+5, from -1 to $+7$
3. x^2-5x-6, from -2 to $+7$
4. $x^2-3x-10$, from -3 to $+6$
5. $x^2-10x+24$, from $+2$ to $+8$
6. $x^2-10x+25$, from $+2$ to $+8$
7. $4x^2-12x+5$, from -2 to $+5$
8. $4x^2+8x-5$, from -4 to $+2$.

9. Show again on each of the graphs the points of the curve where the expression is o. Give the values of x for these points.

216. The equation, $x^2-10x+21=0$, may be regarded as made by placing the expression, $x^2-10x+21$, equal to zero. To solve the equation, $x^2-10x+21=0$, means to find the value or the values of x for which the expression is o. On the graph this means to find how far it is from the o-point, called the origin, to where the curve intersects the horizontal.

The vertical drawn upward through the lowest point, F, of the curve divides the curve into two symmetrical parts. If

the paper were folded over F N as a hinge-line the parts of the curve would coincide.

This line, F N, is called the axis of the curve.

The axis crosses the horizontal at the point $+5$, which is midway between the points $+3$ and $+7$ that represent the two values of x in the equation, $x^2-10x+21=0$.

217. From the values of x given in answering problems 1 to 8 (§215), give exact, or approximate, values of x that will satisfy the following equations:

1. $x^2-8x+12=0$	5. $x^2-10x+24=0$
2. $x^2-6x+5=0$	6. $x^2-10x+25=0$
3. $x^2-5x-6=0$	7. $4x^2-12x+5=0$
4. $x^2-3x-10=0$	8. $4x^2+8x-5=0$.

9. Graph the expression, x^2-4, and from the graph give the values of x that will satisfy the equation, $x^2-4=0$. Check by substituting in the equation.

10. Graph the expression, $10x^2+21x-10$, for values of x between -4 and $+2$ and find from the graph the approximate values of x that satisfy the equation, $10x^2+21x-10=0$.

Quadratic Equations Solved Algebraically

218. The equations that have been solved graphically, sometimes exactly and sometimes only approximately in the foregoing article, may be solved exactly and more easily by algebraic methods. Furthermore, algebraic methods furnish exact solutions for all kinds of quadratic equations.

Exercise LXVI

219. Add a third term to make squares of the following and give the square roots of the resulting trinomials:

1. m^2+n^2	5. m^2+6mn
2. m^2+2mn	6. n^2+8mn
3. $2mn+n^2$	7. x^2+4x
4. m^2+4n^2	8. y^2+6y

9. m^2+10m
10. k^2+3k
11. x^2+5x
12. r^2+7r
13. x^2+bx
14. r^2+3rs
15. $9a^2b^2+4c^2$
16. $9x^2+30k^3x$
17. $4x^2+12x$
18. $4x^2+10x$
19. $16k^2+56ky$
20. $4x^2+11x$.

21. Make a rule for adding a third term to complete the square, (1) when the two square terms are given, (2) when one square term and the cross-product term are given.

220. Solve algebraically for x the equation, $x^2-10x+21=0$.

Write the equation thus:

$$x^2-10x=-21.$$

Add 25 to both sides to make the first side a trinomial square:

$$x^2-10x+25=4, \text{ or}$$
$$(x-5)^2=4.$$

Take the square root of both sides, remembering that 4 has two square roots, $+2$ and -2; thus:

$$x-5=\pm 2.$$

Square-root Axiom.—*Any number has two square roots of the same absolute value, but of contrary sign.*

Using the first sign,

$$x-5=2, \text{ whence } x=+7.$$

Using the second sign,

$$x-5=-2, \text{ whence } x=+3.$$

Check both values of x by substituting in the equation, $x^2-10x+21=0$.

1. Show that $x^2-10x+21=0$ is the algebraic statement of the verbal problem:

The area of a rectangle 10 units long is 21 square units greater than the area of a square whose side equals the unknown dimension of the rectangle.

The algebraic method just given is called the method of solving the equation by **completing the square.**

2. Solve the first six exercises of §**217** by the method of completing the square.

3. Solve the equation, $y^2+5y-36$, by the method of completing the square.

$$y^2+5y+\tfrac{25}{4}=36+\tfrac{25}{4}=\tfrac{169}{4}$$
$$y+\tfrac{5}{2}=\pm\tfrac{13}{2}$$
$$y=\frac{-5\pm 13}{2}=4, \text{ or } -9.$$

4. Show that $y^2+5y-36=0$ is the algebraic statement of the verbal problem:

One dimension of a rectangle is 5 units and the other is equal to the side of a square. The sum of the areas of the rectangle of the square is 36 square units. Find the unknown dimension of the rectangle.

221. Though a quadratic equation generally has *two* solutions, this does not mean that every *problem* that leads to a quadratic equation has two solutions. The nature of the conditions of the problem may be such as to make one, or even both of the solutions of the quadratic impossible, or inappropriate or meaningless. When neither of the two solutions of the quadratic is a solution of the problem it usually means that the conditions of the problem are impossible, or are contradictory, or that the problem is erroneously stated. To decide which solution, if either, meets the conditions stated in a problem, it is necessary to substitute the solutions in the conditions of the problem, and to reject solutions of the equation which do not meet the conditions.

The graphical method used with the quadratic expressions of problems 7 and 8, §**215**, and of problem 10, §**217**, p. 286, furnished only approximate values of x for the corresponding quadratic equations. The algebraic method furnishes exact solutions for quadratic equations.

1. Solve the quadratic equation, $4x^2-12x+5=0$.

By the Div. Ax. the given equation may be written

$$x^2-3x+\tfrac{5}{4}=0.$$

By the Add. Ax. this equation gives

$$x^2-3x+\tfrac{9}{4}=1$$

or, $$(x-\tfrac{3}{2})^2=1.$$

By the Sq.-rt. Ax.

$$x-\tfrac{3}{2}=\pm 1$$

whence, $x=\tfrac{3}{2}\pm 1$

and, $x=\tfrac{5}{2}$, or $\tfrac{1}{2}$.

Check by substituting in the given equation.

2. Solve the equation, $4x^2+8x-5=0$, and check.

3. Solve the equation, $10x^2+21x-10=0$, and check.

4. Solve and check the following quadratics:

(1) $6x^2-17x-14=0$ (4) $4s^2+45s-36=0$

(2) $6x^2+7x-20=0$ (5) $10s^2-21s+10=0$

(3) $9x^2+30x-24=0$ (6) $12s^2-71s+42=0$.

5. Solve the following quadratic equations by the method of completing the square, and check:

(1) $x^2+2x-3=0$

(2) $x^2+4x+3=0$

(3) $x^2+4x-5=0$

(4) $a^2+8a-20=0$

(5) $y^2+14y+45=0$

(6) $t^2+14t+51=0$

(7) $k^2-85=12k$

(8) $z^2=10z+24$

(9) $w^2+91=6w$

(10) $r^2+3r+2=0$

(11) $m^2+5m+6=0$

(12) $h^2+40=13h$

(13) $x^2+x=42$

(14) $x^2+6mx+5m^2=0$

(15) $\dfrac{x^2}{24}-\dfrac{x}{3}=-\tfrac{1}{2}$

(16) $\dfrac{5}{z}=\dfrac{z-1}{z}+\dfrac{z}{z-1}$

(17) $\dfrac{2t+3}{t+8}-\dfrac{2t+9}{3t+4}=0$

(18) $\dfrac{k+1}{k+2}-\dfrac{k+3}{k+4}=\tfrac{8}{3}$.

222. Solve the following problems:

1. The length of a rectangular field is 4 yards more than the width and the area is 60 square yards. Find the dimensions.

2. The hypotenuse of a right triangle is 10 ft. and one of the sides is 2 ft. longer than the other. Find the length of the sides.

3. The sum of the areas of two square fields is 61 square rods, and a side of one is 1 rod longer than a side of the other. Find the sides of both squares.

4. What must be the dimensions of a coal-bin to hold 6 tons of coal, if the depth is 6 ft. and the length is equal to the sum of the width and depth, allowing 40 cu. ft. of space per ton of coal?

5. Telegraph poles are placed at equal distances along a railway. In order that there be two less per mile it would be necessary to increase by 24 ft. the distance between every two consecutive poles. Find the number of poles to the mile.

Quadratics Leading to Irrational Numbers

223. To solve quadratics of the kind to be considered below a new difficulty is met.

1. Graph the expression, m^2+6m+2, between values of m from -8 to $+2$.

m	m^2+6m+2
-8	$+18$
-7	$+9$
-6	$+2$
-5	-3
-4	-6
-3	-7
-2	-6
-1	-3
0	$+2$
$+1$	$+9$
$+2$	$+18$

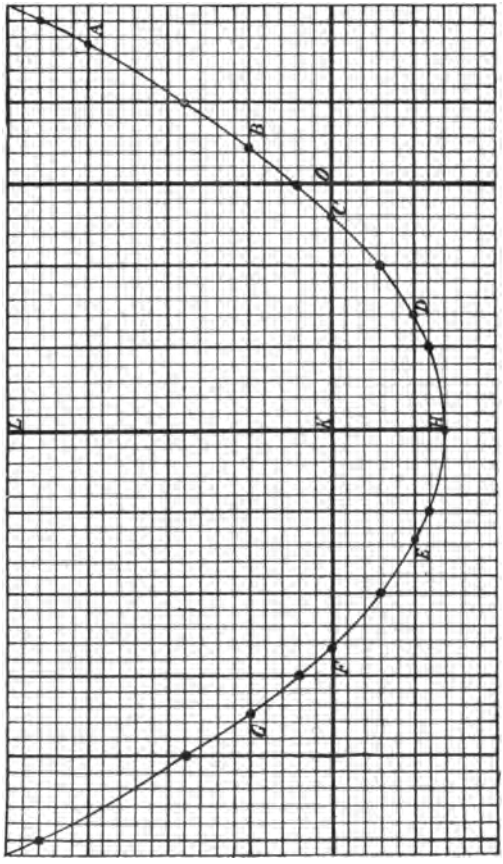


FIG. 239

SCALE.—The unit for m is 1 large square, and for m^2+6m+2, 1 small square.

Plot the points of the table to the indicated scale, and draw through them a smooth curve.

What is the value of m^2+6m+2 at the point A? At B? At C? At D? At H? At E? At F? At G?

What is the value of m at the point A? At B? At C? At D? At H? At E? At F? At G?

The vertical distance of any point of the curve from the the horizontal axis represents the value of the expression, m^2+6m+2, at that point.

What distances on the drawing represent the values of m for any point of the curve?

2. Graph the following quadratic expressions between values of m from -8 to $+2$.

(1) m^2+6m+4 (3) m^2+6m+9
(2) m^2+6m+6 (4) $m^2+6m+10$.

The equation, $m^2+6m+2=0$, is made by placing the expression, m^2+6m+2, equal to zero. Show on the curve of Fig. 239 the values of m for the points where m^2+6m+2 equals zero.

H L is the axis of the curve. It crosses the horizontal at K. Read from the drawing the lengths of K C and K F.

Show that one value of m which gives m^2+6m+2 equal to zero, is

$$\text{O C}=\text{O K}-\text{K C},$$

and that the other value of m is

$$\text{O F}=\text{O K}+\text{K F}.$$

Show from the graph of Fig. 239 that if the two values of m in $m^2+6m+2=0$ be denoted by m' and m'', they may be represented thus,

$$m'=-3+n$$
$$\text{and, } m''=-3-n$$

where n denotes the same number in both values, and (2) that n is some number between 2.5 and 3 units.

224. The algebraic solution of the equation, $m^2+6m+2=0$, will show exactly what this number, n, of **§223**, is, thereby furnishing both values of m exactly.

1. Solve algebraically for m the equation, $m^2+6m+2=0$.

Write $m^2+6m=-2$

Add 9, $m^2+6m+9=7$

or, $(m+3)^2=7$.

Taking square roots, $m+3=\pm\sqrt{7}$.

Whence $m=-3\pm\sqrt{7}$.

The two values of m are then

$$m'=-3+\sqrt{7}$$
$$\text{and } m''=-3-\sqrt{7}.$$

The number, n, used above to denote the distances in Fig. 239 from K to C and from K to F, is $\sqrt{7}$.

2. Solve the following equations algebraically, and point out on the graphs of problem 2 (§**223**) the lines that represent the numbers used in expressing the results:

(1) $m^2+6m+4=0$ (3) $m^2+6m+9=0$

(2) $m^2+6m+6=0$ (4) $m^2+6m+10=0$.

3. Solve the following quadratic equations algebraically:

(1) $x^2+10x+20=0$ (6) $x^2+14x+42=0$

(2) $x^2+10x+22=0$ (7) $x^2+14x+44=0$

(3) $x^2+12x+33=0$ (8) $x^2+16x+62=0$

(4) $x^2+12x+29=0$ (9) $x^2-6x+4=0$

(5) $x^2-14x+41=0$ (10) $x^2-8x+5=0$.

Radicals

225. Numbers like $\sqrt{7}$, $\sqrt{5}$, and $\sqrt{3}$ can be expressed only approximately without using the radical sign. Such numbers are called irrational numbers, or radicals.

In the graphical solution of problem 1, §**223**, one point at which the curve cuts the horizontal is three units to the left of the origin (-3), and back to the right approximately $2\frac{1}{2}$ units, or exactly $\sqrt{7}$ units. That is, one point of intersection is expressed exactly by $-3+\sqrt{7}$.

The second point is three units to the left of the origin (-3), and $2\frac{1}{2}$ additional units approximately ($\sqrt{7}$ units exactly) in the same direction ($-2\frac{1}{2}$, or $-\sqrt{7}$). The second point of intersection is expressed exactly by $-3-\sqrt{7}$.

Point out in Fig. 239 the line that represents $\sqrt{7}$.

Point out on the graph of problem 3 (1) (p. 293) the line that represents $\sqrt{5}$.

Point out the line that represents $\sqrt{3}$ in problem 2 (2) (p. 239).

226. To show in a different way a line whose exact length can be expressed only by a radical, draw a right triangle having a base 3 inches and hypotenuse 4 inches; calculate the length a of the altitude.

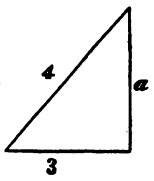


FIG. 240

$$a^2=4^2-3^2=7$$
$$a=\sqrt{7}.$$

Therefore the exact length of the line a is expressed by $\sqrt{7}$, a number that cannot be expressed exactly without the root sign.

Again, if a triangle be constructed with a base of 6 inches, and two sides of 8 inches each, we can draw the altitude and can *calculate* its length, a, thus,

$$a^2=8^2-3^2=55.$$

Therefore, $a=\sqrt{55}$ inches. The number $\sqrt{55}$ cannot, without the root sign, $\sqrt{\ }$, be expressed as a number of units, either integral or fractional.

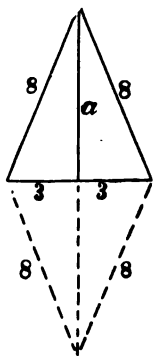


FIG. 241

If another triangle be constructed on this same base with sides of 8 inches extending in the opposite direction from the first, what will be its altitude?

What will be the sum of the two altitudes, or the diagonal of the quadrilateral formed?

This sum is written most briefly thus, $2\sqrt{55}$.

Approximating the Value of a Radical

227. By the process of extracting the square root of an arithmetical number, the square root of 7, i. e., $\sqrt{7}$, is 2.64. But this result is not exact; because the square of 2.64 is only 6.9696, which falls short of 7 by a little more than .03. But the square of 2.65 is 7.0225, a number larger than 7 by .02+. So $\sqrt{7}$ is between 2.64 and 2.65.

```
      7.0000 | 2.64
      4
 46 | 300
      276
524 |  2400
      2096
        304
```

If the root be found to one more decimal place, 2.645 is obtained. The square of this number is 6.996025, which is *less than* 7 by not quite .004. While the square of 2.646 is *larger* than 7 by more than .0003. Then $\sqrt{7}$ lies between 2.645 and 2.646.

Continuing the process of extracting the square root of 7, we obtain 2.6457. The square of this number is *less* than 7 by only .0003, while the square of 2.6458 is *larger* than 7. So $\sqrt{7}$ lies between 2.6457 and 2.6458.

Expressed in symbols:

$$2.64 < \sqrt{7} < 2.65$$
$$2.645 < \sqrt{7} < 2.646$$
$$2.6457 < \sqrt{7} < 2.6458.$$

Thus by continuing the process of extracting the square root of 7 and getting additional figures we may *approximate* $\sqrt{7}$ to any desired degree of accuracy, but we cannot find its exact value. $\sqrt{7}$ expresses the exact value of the square root of 7.

In similar manner show that $\sqrt{55}$ lies between 7.4161 and 7.4162.

228. Indicated roots which can be found only approximately are classed among *irrational numbers*. When written with a radical sign, they are called *radicals*.

Integers and fractions, both positive and negative, are

rational numbers: for example, 2, -5, $\sqrt{4}$, $\sqrt[3]{\frac{1}{8}}$, $\frac{3}{5}$, 3·57, etc.

229. To check the value of m found in problem 1, §**224**, substitute $(-3+\sqrt{7})$ for m in the equation, $m^2+6m+2=0$.

This gives $(-3+\sqrt{7})^2+6(-3+\sqrt{7})+2=0$.

Reducing $(-3+\sqrt{7})^2=(-3)^2+2(-3)(\sqrt{7})+(\sqrt{7})^2$ (1)

$$=9-6\sqrt{7}+7.$$

$(\sqrt{7})^2=7$, because the square root of a number when multiplied by itself (or squared) produces the number. Thus $\sqrt{3}\times\sqrt{3}=3$; $\sqrt{x}\times\sqrt{x}=x$.

The way to express the product of a rational number, as 6, and an irrational number, as $\sqrt{7}$, is to write the rational and irrational numbers side by side thus: $6\sqrt{7}$.

The product, $6\sqrt{7}$, is read: "6 times the square root of 7."

Equation (1) may now be written:

$$9-6\sqrt{7}+7-18+6\sqrt{7}+2=0$$
$$9+7-18+2+6\sqrt{7}-6\sqrt{7}=0 \qquad (2)$$
$$0=0.$$

The equation then is satisfied for the value of $m=-3+\sqrt{7}$.

Show by substituting that the second value of m, $-3-\sqrt{7}$, also satisfies the equation, $m^2+6m+2=0$.

Thus, $(-3-\sqrt{7})^2+6(-3-\sqrt{7})+2=0$.

230. Simplify:

1. $(3+\sqrt{2})^2$
2. $(5+\sqrt{6})^2$
3. $(7-\sqrt{3})^2$
4. $7(3+\sqrt{2})$
5. $4(5-\sqrt{6})$
6. $5(7-\sqrt{3})$.

7. Find the sum of the results of 1 and 4.

Note that $6\sqrt{2}$ added to $6\sqrt{2}$ is $12\sqrt{2}$, the terms being similar with respect to the factor $\sqrt{2}$.

8. Find the sum of the results of 2 and 5; of 3 and 6.

See the note to problem 7.

9. Simplify:

(1) $(-2-\sqrt{2})^2$ (4) $4(-2-\sqrt{2})$

(2) $(-7+\sqrt{8})^2$ (5) $-9(-7+\sqrt{8})$

(3) $(-5-\sqrt{10})^2$ (6) $-3(-5-\sqrt{10})$.

10. Find the sum—

(1) Of (1) and (4) of problem 9

(2) Of (2) and (5) of problem 9

(3) Of (3) and (6) of problem 9.

11. Solve and check the results by substitution:

(1) $r^2+r-7=0$ (5) $m^2-10m=-6$

(2) $d^2+7d+3=0$ (6) $v^2+3v=+11$

(3) $x^2+4x-1=0$ (7) $q^2+11q=+1$

(4) $y^2=6y+2$ (8) $s^2+20s=+28$.

Exercise LXVII

231. Reduce to lowest terms:

1. $\dfrac{3a^2m+2b^2m}{9a^4+12a^2b^2+4b^4}$

2. $\dfrac{25c^2+10cd+d^2}{5ac+ad+5bc+bd}$

3. $\dfrac{2ax-6x-2ay+6y}{3akx-9kx-3aky+9ky}$

4. $\dfrac{10klx^4+15klx^2+20kl}{2x^5+2x^4+3x^3+3x^2+4x+4}$.

232. Perform the indicated operations:

5. $\dfrac{5}{3x+6}-\dfrac{7}{x^2+4x+4}$

6. $\dfrac{x^2-xy}{x^2-2xy+y^2}\cdot\dfrac{x-y}{x^2+xy}$

7. $\dfrac{6}{10x-15}+\dfrac{7}{4x-6}-\dfrac{x-4}{4x^2-12x+9}$

8. $\dfrac{4t^2+4t+1}{3t+5}\cdot\dfrac{3ct+5c+6dt+10d}{2ct+c+4dt+2d}$

9. $\dfrac{64a^2-96a^2b+35ab^2}{4a^2-6ab}\div\dfrac{16a^2-12ab}{8a-72a^2b+16a}$

The Difference of Two Squares

Exercise LXVIII

233. Multiply:

1. $(a+b)(a-b)$
2. $(a+y)(x-y)$
3. $(m-n)(m+n)$
4. $(r-s)(r+s)$
5. $(u+v)(u-v)$
6. $(x^2+y)(x^2-y)$
7. $(x+y^3)(x-y^3)$
8. $(a^2-y^2)(a^2+y^2)$
9. $(2x+y)(2x-y)$
10. $(a+5b)(a-5b)$
11. $(3m-2n)(3m+2n)$
12. $(ab+c)(ab-c)$
13. $(m^2n+x^3)(m^2n-x^3)$
14. $(2u^2v-w)(2u^2v+w)$
15. $(3a^2b-2c^4)(3a^2b+2c^4)$
16. $(4abc^2-3x^5)(4abc^2+3x^5)$.

17. From these problems make a rule for multiplying the sum of two numbers by their difference.

18. Make a rule for factoring the difference of two squares.

Exercise LXIX

Factor the following expressions and test the results, doing all you can mentally:

1. x^2-y^2
2. $16k^2-25b^2$
3. $49a^2-9b^2$
4. $m^2n^2-144r^2$
5. $289u^2-81v^2$
6. $16-25y^2$
7. $169d^2h^2s^2-225t^4$
8. $9m^4-121n^2q^4$
9. $49a^2-100b^4c^6$
10. $196-361a^4k^6x^2$
11. ax^4-100a
12. p^2q^2r-r
13. $225b^{10}-f^4g^6h^{14}$
14. $4a^2cd-25c^5d$
15. r^4-s^4
16. $81m^4-16n^4r^4$
17. kx^8-ky^8
18. $256a^4-625c^4d^8$
19. m^6-n^6
20. $a^{10}-b^{10}$
21. $(r+3s)^2-16t^2$
22. $(2a+b^2)^2-9c^4$
23. $(5x^2-3y^3)^2-16z^4$
24. $(x^2-y)^2-x^6$
25. $b^2-(3a+2c)^2$
26. $16a^2-(2m-3n)^2$

27. $9c^2-(2a-3c)^2$
28. $(3a-2b)^2-(2c-3d)^2$
29. $(4a+5)^2-(2x-3)^2$
30. $36(a+b)^2-25(c-d)^2$
31. $(a+b+c)^2-(a-b-c)^2$
32. $a^2-2ay+b^2-c^2$
33. $x^2-6xy+9y^2-16z^2$
34. $25x^2+16y^2-4a^2+40xy$
35. $k^2-x^2-2xy-y^2$
36. $1-a^2-2ab-b^2$
37. $9m^2-a^2-4ab-4b^2$
38. $36r^2-4+20t-25t^2$
39. $x^2+2xy+y^2-a^2-2ab-b^2$
40. $a^2+2a+2bc-b^2-c^2+1$
41. $9x^2+16y^2-49a^2-4b^2+28ab+24xy$
42. $9a^2-12ab+4b^2-16x^2-8xy-y^2$
43. $x^4+x^2y^2+y^4$

Suggestion:

$$x^4+x^2y^2+y^4=x^4+2x^2y^2+y^4-x^2y^2=(x^2+y^2)^2-x^2y^2.$$

44. $a^4-7a^2b^2+b^4$
45. x^4+x^2+1
46. $16x^4-17x^2y^2+y^4$
47. $25x^4+31x^2y^2+16y^4$
48. $a^2x^8+a^2x^4+a^2$
49. $49a^4b^4-53a^2b^2x^2+4x^4$.

50. Give mentally the product: $78\cdot82$.

$$78\cdot82=(80-2)(80+2)=80^2-2^2=6{,}400-4=6{,}396.$$

51. Give mentally the following products:

(1) $19\cdot21$	(6) $27\cdot33$	(11) $88\cdot92$
(2) $18\cdot22$	(7) $26\cdot34$	(12) $65\cdot75$
(3) $17\cdot23$	(8) $39\cdot41$	(13) $87\cdot93$
(4) $29\cdot31$	(9) $38\cdot42$	(14) $85\cdot95$
(5) $28\cdot32$	(10) $67\cdot73$	(15) $98\cdot102$.

Exercise LXX

234. Reduce the following fractions to lowest terms:

1. $\dfrac{mx^2-my^2}{ax-ay+bx-by}$
2. $\dfrac{x^2+ax+bx+ab}{(x^2-a^2)(x^2-b^2)}$
3. $\dfrac{k^3m^4+k^3m^2n^2+k^3n^4}{a^2m^2+a^2mn+a^2n^2}$
4. $\dfrac{12a^4m^2+48a^4-1}{36a^8-60a}$

5. $\dfrac{(4a-d)^2-(2b-3c)^2}{(4a-2b)^2-(3c-d)^2}$

6. $\dfrac{a^2+4ab+4b^2-9c^2}{a^2+4b^2+9c^2+4ab-6ac-12bc}$

7. $\dfrac{1-a^2-2ab-b^2}{1+a^2+b^2+2a+2b+2ab}$.

235. In the following exercises perform the indicated operations:

1. $\dfrac{a^2+16a+64}{a^2-8a+16}\cdot\dfrac{a^2-16}{a^2-64}$

2. $\dfrac{4x^2-12xy+9y^2}{x^2-a^2-2ab-b^2}\cdot\dfrac{x^2+a^2+b^2+2ax+2bx+2ab}{4ax^2-9ay^2}$

3. $\dfrac{a^5+ab^4}{12a^2-75b^4}\div\dfrac{a^3+a^2b+ab^2+b^3}{24a^2b-120ab^3+150b^5}$

4. $\dfrac{2abx+3aby+2a^2b^2x+3a^2b^2y}{x^3+2abx^3+a^2b^2x^3}\div\dfrac{4x^2+12xy+9y^2}{mx^3+nx^2}$

5. $\dfrac{a^4-b^4}{a^2k^2+2abk^2+b^2k^2}\cdot\dfrac{mr+ms}{a^2-2ab+b^2}\div\dfrac{mc+mb}{k^2}$

6. $\dfrac{x}{2x-3}-\dfrac{x}{2x+3}-\dfrac{2x-6}{4x^2-9}$

7. $\dfrac{2r-1}{r-1}+\dfrac{4}{r+1}-\dfrac{3r}{r^2-1}-2$.

Exercise LXXI

236. Solve the following equations:

1. $ak+bk=a^2-b^2$. Solve for k
2. $ax+3bx=a^2+6ab+9b^2$. Solve for x
3. $m^2-10m=75$. Solve for m
4. $a^2m-b^2m=an+bn$. Solve for m; solve for n
5. $r^2=7r+44$. Solve for r
6. $3ut+6vh^3+5wt=9uh^3+2vt+15wh^3$. Solve for t

7. $m+n+\frac{n^2}{x}=\frac{m^2}{x}$. Solve for x

8. $\frac{x}{b^2}-\frac{4x}{a^2}=\frac{1}{b^2}-\frac{4}{ab}+\frac{4}{a^2}$. Solve for x

9. $\frac{m}{8}=1+\frac{6}{m}$. Solve for m

10. $\frac{u^2}{t}=\frac{u}{g}+\frac{v^2}{t}+\frac{v}{g}$. Solve for g; for t.

Trinomials of the Form x^2+ax+b

237. Multiply:

1. $(x+3)$ by $(x+4)$
2. $(x+7)$ by $(x-2)$
3. $(x-5)$ by $(x+3)$
4. $(x-3)$ by $(x-8)$
5. $(x+5)$ by $(x+11)$
6. $(x+9)$ by $(x-8)$
7. $(x-10)$ by $(x+7)$
8. $(x-2)$ by $(x-13)$.

9. From these cases make a rule for multiplications of this type.

10. The general law thus obtained is proved as follows:

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \quad\;\; \underline{+bx+ab} \\ x^2+ax+bx+ab=x^2+(a+b)x+ab. \end{array}$$

Therefore $x^2+(a+b)x+ab=(x+a)(x+b)$.

The sum of $+a$ and $+b=a+b$, the coefficient of x in the trinomial. The product of $+a$ and $+b=+ab$, the term of the trinomial not containing x (i. e., the absolute term). This gives us a method of factoring by inspection.

11. Factor $x^2+12x+35$ by inspection.

Find two factors of $+35$ whose sum is $+12$. The factors are $+5$ and $+7$. Then the factors of $x^2+12x+35$ are $(x+5)$ and $(x+7)$.

12. Factor $x^2-3x-40$ by inspection.

The factors of -40 whose sum is -3, are -8 and $+5$. Therefore $x^2-3x-40=(x-8)(x+5)$.

EXERCISE LXXII

238. Factor by inspection the following expressions:

1. x^2+3x+2
2. m^2+5m+6
3. $k^2+12kb+35b^2$
4. $y^2+14y+45$
5. $h^2-13h+40$
6. $a^2+8a-20$
7. $g^2+3g-180$
8. $b^2+19bc+48c^2$
9. q^6+q^3-42
10. $t^2-14t-51$
11. $r^2-2rs-323s^2$
12. $v^2-6vw-91w^2$
13. $a^2b^2-4abc^2-165c^4$
14. $z^4-10z-24$
15. $y^2-12y-85$
16. $d^2-7d-30$
17. p^2-p-90
18. $r^2-13r-48$
19. $k^2+40kb+111b^2$
20. $x^2y^2-9xyz-112z^2$
21. $r^{2m}-19r^m+48$
22. $k^{6t}-7k^{3t}-78$
23. $d^{10r}+23d^{5r}f^{3t}+102f^{6t}$
24. $k^{2m}-20k^mL^r.-69L^{2r}$
25. $r^2s^2t^6-18rst^3v^2w-63v^4w^2$
26. $(m+n)^2+5(m+n)+6$
27. $(a+b)^2-7(a+b)+10$
28. $a^2+2ab+b^2+3a+3b-10$
29. $a^2-6ab+9b^2+7ac-21bc-44c^2$
30. $m^4x+m^2x-650x$.

Quadratic Equations Solved by Factoring

239. Solve the equation $x^2-8x+12=0$.

$$x^2-8x+12=0$$

Factoring

$$(x-2)(x-6)=0 \qquad (1)$$

This equation is satisfied if

$$x-2=0 \qquad (2)$$

because—

The product of two or more numbers is zero if, and only if, at least one of the numbers is zero.

From equation (2) $x=2$.

CHECK: $2^2-8\cdot 2+12=4-16+12=0$.

Likewise from (1) $x-6=0$,

Whence, $x=6$. Check.

Consequently, both 2 and 6 are roots of the equation.

1. Solve the same problem by completing the square. (See pp. 287, 288.)

2. Solve the equation $x^2=x+42$.

$$x^2-x-42=0$$
$$(x-7)(x+6)=0$$
$$x=7, \text{ or } -6.$$

State the principle used in the last step. Test both results.

Solve the same problem by the method of completing the square.

EXERCISE LXXIII

240. Solve the following quadratic equations by the method of factoring, and test results:

1. $x^2-3x+2=0$
2. $y^2-4y+3=0$
3. $m^2=m+2$
4. $n^2+5n=6$
5. $a^2+7a+6=0$
6. $m^2=4m+12$
7. $k^2+k=56$
8. $r^2+51=20r$
9. $b^2=4b+77$
10. $c^2+112=23c$
11. $\frac{x^2}{15}=\frac{x}{5}+\frac{2}{3}$
12. $\frac{x^2}{10}-3=\frac{13x}{10}$
13. $\frac{x^2}{21}-\frac{4x}{7}=\frac{4}{3}$
14. $\frac{x^2}{6}=\frac{x}{2}+9$
15. $\frac{4x}{5}=\frac{x^2}{15}+\frac{7}{3}$
16. $\frac{x}{2}+\frac{15}{7}=\frac{x^2}{14}$
17. $\frac{x^2}{13}+\frac{6x}{13}=7$
18. $\frac{4x}{x-1}+\frac{x-10}{x}=4$
19. $\frac{5y-1}{9}+\frac{3y-1}{5}=\frac{10y}{9}-\frac{4}{9y}$
20. $\frac{a+7}{9-4a^2}=\frac{1-a}{2a+3}+\frac{4}{2a-3}$.

Exercise LXXIV

241. Solve the following problems:

1. The base of a triangle exceeds the altitude by 4 inches, and the area is 30 square inches. Find the base and altitude.

2. A rectangular field is twice as long as wide. If it were 20 rods longer and 24 rods wider, the area would be doubled. What are the dimensions?

3. The perimeter of a rectangular field is 60 rods. The area is 200 square rods. Find the dimensions.

4. A tree standing on level ground was broken over so that the top touched the ground 50 feet from the stump. The stump was 20 feet more than two-fifths of the height of the tree. What was the height of the tree?

5. A can do a piece of work in 3 days less than B; and both can do the work in 2 days. How long will it take each alone?

Forming Quadratic Equations Having Given Roots

242. By reversing the steps of the solution of a quadratic equation, an equation may be formed having given roots.

1. Examine again the solution of the equation, $x^2-8x+12=0$, §**239**.

2. Form a quadratic equation, whose roots are 2 and 6.

$x-2=0$ is an equation having the root 2
$x-6=0$ is an equation having the root 6
and $(x-2)(x-6)=0$ is an equation having the roots 2 and 6.

Therefore—

$x^2-8x+12=0$ is the required equation.
Show by substitution that 2 and 6 are roots of the last equation.
Why is the equation a quadratic equation?

3. Form quadratic equations having the following pairs of roots:

(1)	2 and 4	(6)	$-\frac{1}{2}$ and -6
(2)	2 and -5	(7)	-8 and 9
(3)	-3 and 5	(8)	7 and -9
(4)	-3 and -5	(9)	5 and -12
(5)	$\frac{1}{2}$ and 4	(10)	a and b.

243. Exercise (10), which gives the quadratic equation $x^2-(a+b)x+ab=0$, shows that—

The coefficient of the first power of the unknown is the sum of the required roots, taken negatively, and the absolute term is the product of the roots.

1. Applying the law just stated, form a quadratic equation whose roots are $2+\sqrt{2}$ and $2-\sqrt{2}$.

The sum of the roots is $2+\sqrt{2}+2-\sqrt{2}=4$

and the product is $(2+\sqrt{2})(2-\sqrt{2})=4-(\sqrt{2})^2=4-2=2.$

(See §229.)

The required quadratic equation is then

$$x^2-4x+2=0. \qquad (1)$$

2. Show by substituting that $2+\sqrt{2}$ and $2-\sqrt{2}$ are roots of equation (1).

3. Form quadratic equations having the following pairs of roots:

(1)	$2+\sqrt{3}$ and $2-\sqrt{3}$	(6)	$-3+\sqrt{3}$ and $-3-\sqrt{3}$
(2)	$3+\sqrt{3}$ and $3-\sqrt{3}$	(7)	$-3+\sqrt{5}$ and $-3-\sqrt{5}$
(3)	$3+\sqrt{5}$ and $3-\sqrt{5}$	(8)	$-4+\sqrt{5}$ and $-4-\sqrt{5}$
(4)	$4+\sqrt{7}$ and $4-\sqrt{7}$	(9)	$-5+\sqrt{7}$ and $-5-\sqrt{7}$
(5)	$5+\sqrt{10}$ and $5-\sqrt{10}$	(10)	$-6+\sqrt{21}$ and $-6-\sqrt{21}$.

Exercise LXXV

244. Factor the following by any method and test results:

1. $a^2bc^3-2abc^3-8bc^3$
2. $b^2m+b^2n-12bm-12bn-45m-45n$
3. p^4-26p^2+25
4. p^4-16

5. $2ax+2ay-2az+2bx+2by-2bz$
6. $v^{16}-w^{16}$ (into five factors)
7. $m^2-2mn+n^2-c^2-d^2-2cd$
8. $8x^3-6x^2-28x+21$
9. $3k^2-6kl+3l^2+6km-6lm$
10. $x-4y+12yz-9z$
11. $(m+n)^2-11(m+n)(c+d)+28(c+d)^2$
12. $abc^2x-10abcx-75abx$
13. $810a^4c^3-10b^4c^3$
14. $a^{2m}-5a^mb^{3n}-36b^{6n}$
15. m^4+4n^4.

245. Reduce the following fractions to lowest terms:

1. $\dfrac{x^2-3x+2}{x^2-4x+3}$
2. $\dfrac{x^2+ax+bx+ab}{x^2+3ax+2a^2}$
3. $\dfrac{2ax+3bx+4a+6b}{x^2+x(b+2)+2b}$
4. $\dfrac{2x+bx-3b-6}{ax-3a+bx-3b}$
5. $\dfrac{m^2-12m+32}{m^2+2m-24}$
6. $\dfrac{x^{2n}-4x^n+3}{x^{2n}-6x^n+5}$.

246. Perform the indicated operations in the following problems:

1. $\dfrac{a^2-2a-35}{2a^3-3a^2}\cdot\dfrac{4a^3-9a}{a-7}$
2. $\dfrac{1}{m^2-7m+12}-\dfrac{2}{m^2-4m+3}-\dfrac{3}{m^2-5m+4}$
3. $\dfrac{y-1}{y^2-7y+10}-\dfrac{y+2}{y^2-9y+14}-\dfrac{y-3}{y^2-12y+35}$
4. $\left(2a-\dfrac{a^2-b^2}{a}\right)\left(3b+\dfrac{a^2+b^2}{b}\right)\div\left(\dfrac{a^2}{b^2}+5+\dfrac{4b^2}{a^2}\right)$
5. $\dfrac{x^2+y^2+2xy-z^2}{z^2-x^2-y^2+2xy}\div\dfrac{x+y+z}{y+z-x}$.

Trinomials of the Form ax^2+bx+c

247. Multiply:

1. $(3x+5)(2x+3)$
2. $(5x+4)(3x+2)$
3. $(3x-5)(2x+3)$
4. $(5x+4)(3x-2)$
5. $(2x-7)(4x+3)$
6. $(3x-5)(2x-3)$
7. $(5x-4)(3x-2)$
8. $(2x-7)(4x-3)$.

9. Make a rule for multiplying binomials of this type.

The general law thus obtained is proved as follows:

$$\begin{array}{l} ax+b \\ \underline{cx+d} \\ acx^2+bcx \\ \underline{\qquad adx+bd} \\ acx^2+(bc+ad)x+bd. \end{array}$$

The product of a and c is the coefficient of x^2, the product of b and d is the absolute term (the term not containing x), the sum of the cross products bc and ad, that is, $(bc+ad)$ is the coefficient of x. This gives a method of factoring by inspection.

10. Factor $3x^2+17x+10$.

Find factors of 3 and factors of 10 such that the sum of the cross products is 17.

The various possibilities are:

$$\begin{array}{c} +3\times+10 \\ \underline{+1\quad+1} \\ +13 \end{array} \quad \begin{array}{c} +3\times+1 \\ \underline{+1\quad+10} \\ +31 \end{array} \quad \begin{array}{c} +3\times+5 \\ \underline{+1\quad+2} \\ +11 \end{array} \quad \begin{array}{c} +3\times+2 \\ \underline{+1\quad+5} \\ +17 \end{array}$$

+13 cross product. +31 cross product. +11 cross product. +17 cross product.

Then $3x^2+17x+10=(3x+2)(x+5)$. Check.

11. Factor $8y^2-37yz-15z^2$.

Since the middle term is negative, there are the following possibilities:

$$\begin{array}{c} +2\times-15 \\ \underline{+4\quad+1} \\ -58 \end{array} \quad \begin{array}{c} +2\times+1 \\ \underline{+4\quad-15} \\ -26 \end{array} \quad \begin{array}{c} +2\times-5 \\ \underline{+4\quad+3} \\ -14 \end{array} \quad \begin{array}{c} 2\times-3 \\ \underline{4\quad+5} \\ -2 \end{array} \quad \begin{array}{c} +8\times-15 \\ \underline{+1\quad+1} \\ -7 \end{array} \quad \begin{array}{c} +8\times+1 \\ \underline{+1\quad-15} \\ -119 \end{array}$$

$$\begin{array}{c} +8\times+5 \\ \underline{+1\quad-3} \\ -19 \end{array} \quad \begin{array}{c} +8\times+3 \\ \underline{+1\quad-5} \\ -37 \end{array}$$

Then $8y^2-37yz-15z^2=(8y+3z)(y-5z)$. Check.

Exercise LXXVI

248. Factor the following expressions, and test the results:

1. $2x^2+11x+12$
2. $8c^2+46c-12$
3. $3x^2-17x+10$
4. $8z^2-31z+21$
5. $5x^2-38x+21$
6. $11a^2-23ab+2b^2$
7. $7k^2+123k-54$
8. $12t^2+31st-15s^2$
9. $5m^2-29mn+36n^2$
10. $10r^2-23r-5$
11. $6b^2-29b+35$
12. $6f^2-f-77$
13. $102-11a-a^2$
14. $15+37z-8z^2$
15. $1-6xy+5x^2y^2$
16. $2x^{2n}+11x^n+12$
17. $14x^{2m}+53x^my^n+14y^{2n}$
18. $6(a+b)^2-7(a+b)-3$
19. $4x^2+8xy+4y^2+13x+13y+3$
20. $3c^2-6cd+3d^2-2c+2d-5$.

249. Factor the following by any method and test the results:

1. $2c^2xy-13cdxy+6d^2xy$
2. $3x^3+2x^2-9x-6$
3. x^4-2x^2+1
4. $6a^3-30a^2b+36ab^2$
5. $a^2-b^2-c^2-2bc+a+b+c$
6. a^2-b^2+a-b
7. $16x^4-81$
8. a^9-256a
9. $4a^4-9a^2+6a-1$
10. $a^2b^3-4ab^3-77b^3$
11. $66+39m^3+3m^6$
12. $b^{2n}-24b^n+63$
13. $a^2-a+\frac{1}{4}$
14. $m^2-\frac{4m}{3n}+\frac{4}{9n^2}$
15. $7r^2-.07$
16. $a^2t^2+b^2r-b^2s+a^2r+b^2t^2-a^2s$.

Exercise LXXVII

250. Reduce the following fractions to lowest terms:

1. $\dfrac{2m^2+5m+2}{2m^2+7m+3}$
2. $\dfrac{2x^2+5x+3}{3x^2+7x+4}$

3. $\dfrac{2a^2+17a+21}{3a^2+26a+35}$

4. $\dfrac{6d^2-5d-6}{8d^2-2d-15}$

5. $\dfrac{am^2-am-20a}{2m^2-7m-15}$

6. $\dfrac{15k^2+kl-21}{9k^2+3kl-21}$

7. $\dfrac{y^2z-8yz+15z}{2ay^2-13ay+21a}$

8. $\dfrac{6x^2-bx-12b}{9x^2-3bx-12b}$.

251. In the following problems perform the indicated operations:

1. $\dfrac{b^2+b-6}{2b^2+5b-3}-\dfrac{b^2+8b+15}{2b^2+7b-15}$

2. $\dfrac{3}{18c^2+c-5}-\dfrac{4}{2c^2+7c-4}$

3. $\dfrac{10bx+3b^2+3x^2}{10bx-3b^2-3x^2}\div\dfrac{(3b+x)x}{(x-3b)b}$

4. $\dfrac{y^2+y-1}{2y^2-y-3}-\dfrac{y^2-y-1}{3y^2-y-4}$

5. $\dfrac{y^2-14y+24}{w^2+9w-36}\div\dfrac{(y^2+4y-12)(yw+5y)}{(w^2+2w-15)(yw+6w)}$.

252. Solve the following first degree equations for the desired number.

1. $2am^2-5am+3m=bn^2+bn-2b$. Solve for a; for b

2. $15kx^2+kxy-6ky=9x^2+3xy-2y$. Solve for k

3. $tx^2-lx=20t+2x^2-7x-15$. Solve for t

4. $2a^2x+17ax+21x-3a^2y-35y=0$. Solve for x; for y

5. $mx^2+15m+13mx=8mx+2nx^2+21n$. Solve for m; for n.

253. Solve the quadratic equation, $2x^2-13x+6=0$.

Factoring, $(2x-1)(x-6)=0$.

Then, by the principle stated in §**239**, the equation is satisfied if $2x-1=0$; whence $x=\frac{1}{2}$.

The equation is also satisfied if $x-6=0$; whence $x=6$.

The two solutions of the quadratic equation, $2x^2-13x+6=0$, are $x=\frac{1}{2}$ and $x=6$.

Check both results.

254. Solve the following quadratic equations:

1. $2x^2-5x+2=0$
2. $6x^2+17x+12=0$
3. $6x^2=mx+35m$. Solve for m; for x
4. $36k^2=19kl+6l$. Solve for k; for l
5. $5y^2+13y=6$
6. $24m^2+15n^2=38mn$. Solve for m; for n
7. $2c^2+6d^2=13cd$. Solve for c; for d
8. $8a^2+14ab=15b^2$. Solve for a; for b
9. $14x^2+21y^2=55xy$. Solve for x; for y
10. $8k^2l^2=22klm+21m^2$. Solve for kl, for k, and m
11. $\frac{a^2}{7}-\frac{a}{2}=\frac{-3}{14}$
12. $\frac{1}{x-2}-\frac{2}{x+2}=\frac{3}{5}$
13. $\frac{m}{m+1}+\frac{m+1}{m}=\frac{13}{6}$
14. $\frac{14}{y+4}-\frac{5}{y+2}=\frac{3}{y}$
15. $\frac{1}{x-1}+\frac{x-1}{x^2-4x+4}=\frac{2x-5}{2x^2-6x+4}$.

255. Solve the following problems:

1. The length of a rectangle is 5 greater than the width, and the area is $18\frac{3}{4}$. Find the dimensions.

2. B can do a piece of work in 3 less days than A, and both can do the work in $5\frac{1}{7}$ days. How long will it take each alone?

3. A walks faster than B by a quarter of a mile in an hour. It takes A a quarter of an hour less than it takes B to walk 15 miles. Find the rate at which each walks.

4. A freight train takes $1\frac{1}{2}$ hours longer to travel 100 miles than a passenger train takes. The passenger train runs 10

miles per hour less than twice the rate of the freight train. Find the rate of each train.

5. In three hours a boatman rowed 10 miles up a stream and 4 miles back. If the velocity of the current was 2 miles an hour what was his rate of rowing?

The Sum or the Difference of Two Cubes

256. Multiply:

1. $(x+y)(x^2-xy+y^2)$
2. $(x-y)(x^2+xy+y^2)$
3. $(a+b)(a^2-ab+b^2)$
4. $(a-b)(a^2+ab+b^2)$
5. $(a+2b)(a^2-2ab+4b^2)$
6. $(3a-b)(9a^2+3ab+b^2)$
7. $(2a+3b)(4a^2-6ab+9b^2)$
8. $(3a^2+5b^2)(9a^4-15a^2b^2+25b^4)$
9. $(7a^3-4b^2)(49a^6+28a^3b^2+16b^4)$
10. $(2a^nb^n-3c^n)(4a^{2n}b^{2n}+6a^nb^nc^n+9c^{2n})$.

11. Make a rule for multiplying the sum of two numbers by the square of the first number minus the product of the first and second numbers plus the square of the second number.

12. Make a rule for multiplying the difference of two numbers by the square of the first number plus the product of the first and second numbers plus the square of the second number.

13. Make a rule for factoring the sum of two cubes.

14. Make a rule for factoring the difference of two cubes.

15. Factor $64a^3+27b^3$.

$$64a^3+27b^3=(4a+3b)(16a^2-12ab+9b^2).$$

The expression is the sum of two cubes $(4a)^3$ and $(3b)^3$. Therefore one factor is the sum of the cubes of the numbers ($4a$ and $3b$). The other factor is the square of the first number of the first factor, $(4a)^2=16a^2$, minus the product of the first and second numbers, $-(4a\times 3b)=-12ab$, plus the square of the second number $(3b)^2=9b^2$. Check the result.

16. Factor $8x^3-125y^3$.

$8x^3-125y^3=(2x-5y)(4x^2+10xy+25y^2)$. Check the result.

257. Factor the following expressions, doing as many as you can mentally:

1. a^3+b^3
2. $8x^3-y^3$
3. m^3+27n^3
4. $8c^3-d^3$
5. $343-x^3$
6. t^3+64
7. $8x^{18}+27z^{12}$
8. $27u^9w^6+1$
9. $8v^{18}+27w^{18}$
10. $729a^6+216c^6$
11. $512c^3-27d^3$
12. k^3l^3+343
13. $a^{3m}+b^{3m}$
14. $8x^{3r}-y^{3r}$
15. $27m^{3a}+8n^{6b}$
16. $(a+b)^3+c^3$
17. $(a+m)^3-n^3$
18. $a^3+(b+c)^3$
19. $(w+3)^3-a^3$
20. $(5m-n)^3+c^3$.

The Remainder Theorem

1. What is the remainder after dividing $3x+7$ by $x-1$? What is the value of $3x+7$ if 1 is substituted for x?

2. What is the remainder after dividing $4x+5$ by $x-2$? What is the value of $4x+5$ if 2 is substituted for x?

3. What is the remainder after dividing $3x-6$ by $x-2$? What is the value of $3x-6$, if $x=2$?

258. In the following problems give the remainders after dividing and evaluate as required:

1. Divide $4x-3$ by $x-3$. Evaluate the dividend for $x=3$.

2. Divide $11x-13$ by $x-10$. Evaluate the dividend for $x=10$.

3. Divide $2x+3$ by $x+1$. Evaluate the dividend for $x=-1$.

4. Divide $11x+25$ by $x+2$. Evaluate the dividend for $x=-2$.

5. Divide $20x-37$ by $x+5$. Evaluate the dividend for $x=-5$.

6. Divide $ax+b$ by $x-r$. Evaluate the dividend for $x=r$.

7. Divide $my-n$ by $y-c$. Evaluate the dividend for $y=c$.

8. Divide $ax+b$ by $x+r$. Evaluate the dividend for $x=-r$.

9. Divide $mv+n$ by $v+t$. Evaluate the dividend for $v=-t$.

10. Divide $x^3+5x^2-6x-11$ by $x-2$. Evaluate the dividend for $x=2$.

11. When an expression in x is to be divided by x plus a number, or x minus a number, how may the remainder be obtained without dividing?

In the following problems give the remainders without dividing and check the results by division:

1. $(3x-7)\div(x-1)$
2. $(7x+3)\div(x+1)$
3. $(3x+11)\div(x+2)$
4. $(x^2+11)\div(x+3)$
5. $(x^2+5x-36)\div(x-4)$
6. $(2x^3+x^2-5x+7)\div(x+8)$.

7. Divide ax^2+bx+c by $x-r$.

$$
\begin{array}{l}
x-r)\,ax^2+bx+c\,(ax+b+ar \\
\quad\;\; \underline{ax^2-axr} \\
\qquad\quad bx+axr \\
\qquad\quad \underline{bx-br} \\
\qquad\qquad axr+br+c \\
\qquad\qquad \underline{axr-ar^2} \\
\qquad\quad \text{Remainder, } ar^2+br+c.
\end{array}
$$

Obtain the remainder by the rule given in problem 11.

8. Without dividing, give the remainder after dividing kx^3+lx^2+mx+n by $x-1$. Check by division.

9. Without dividing, give the remainder after dividing x^2+px+q by $x+k$. Check by division.

259. If in a division problem the remainder is zero, the division is exact, and the divisor is a factor of the dividend.

260. Determine, without dividing, whether the first expression is a factor of the second in the following, and factor the factorable expressions:

1. $x-2\ldots\ldots 3x-6$
2. $x-1\ldots\ldots x^2-3x+2$

3. $x+1 \ldots\ldots x^2-12x-13$

4. $x-2 \ldots\ldots x^3-x^2+3x+2$

5. $x+7 \ldots\ldots x^3+5x^2-29x-105$.

261. The proof will now be given of the theorem we have been using, i. e., that, if in a given expression containing x (call it for brevity an expression in x), r is put for x, the expression in x reduces to the remainder R, that is obtained by dividing the expression in x by $x-r$.

In division there is always a divisor d, a dividend D, a quotient Q, and a remainder R (sometimes zero), and the following relation connects them:

$$D=Q\times d+R.$$

Then, if the expression in x is divided by $x-r$, by this principle we have the

$$\text{expression in } x=Q(x-r)+R.$$

Substitute on both sides, for x, the number r, then the expression in $r=Q(r-r)+R=Q\times 0+R=R$.

Therefore, R, the remainder found by dividing the expression in x by $x-r$, equals the expression in r, i. e., the given expression with r put in place of x.

This is called the **remainder theorem**.

262. Multiply the following expressions and make a rule for finding the constant term in the product from the constant terms of the factors:

1. $(x+4)(x+5)$
2. $(x-3)(x+7)$
3. $(x-2)(x-6)$
4. $(x+2)(x^2+5)$
5. $(x+3)(x^2+x-4)$
6. $(x-5)(2x^2+x-8)$
7. $(x+2)(x+3)(x+4)$
8. $(x+1)(x+3)(x-5)$
9. $(x+2)(x-3)(x-2)$
10. $(x-3)(x-4)(x-5)$.

11. From problems 1–10 it is clear that if an expression with integral coefficients has a factor $x-a$, then a is a factor of the constant or absolute term of the expression.

12. Factor by the *remainder theorem*, $x^3-7x^2+16x-12$.

The exact divisors of -12 are $1, -1, 2, -2, 3, -3, 4, -4, 6, -6$, and $12, -12$.

For $x=1$, the value of $x^3-7x^2+16x-12$ is -2.

For $x=-1$, the value is -36.

For $x=2$, the value is 0.

Therefore, $x-2$ is *a factor* of $x^3-7x^2+16x-12$.

By division, the other factor is x^2-5x+6.

Then, $x^3-7x^2+16x-12=(x-2)(x-2)(x-3)$.

263. Factor by the remainder theorem:

1. x^3-1
2. $x^3-7x^2+7x+15$
3. $x^3-5x^2-2x+24$
4. x^3+8
5. $x^3+2ax^2+5a^2x+4a^3$
6. y^3+64
7. $r^3+r^2-22r-40$
8. $2m^3+9m^2+12m+4$
9. $x^4-3x^3-21x^2+43x+60$
10. $x^4-15x^2+10x+24$.

264. Solve $x^3+3x^2-13x-15=0$.

$x=-1$, reduces the expression on the left to zero, then $x-(-1)$, or $x+1$ is a factor.

By dividing we get:

$$\begin{aligned} x^3+3x^2-13x-15 &= (x+1)(x^2+2x-15) \\ &= (x+1)(x-3)(x+5). \end{aligned}$$

Then, $(x+1)(x-3)(x+5)=0$,

Whence, $x=-1, 3, -5$. Why? Check.

265. Solve the following equations:

1. $m^3-6m^2+11m-6=0$
2. $x^3+2x^2=11x+12$
3. $4x+16=x^3+4x^2$
4. $x^3+5x^2-45=9x$
5. $y^3+120=7y^2+14y$
6. $k^3+2k^2-4k-8=0$
7. $x^3=5x^2-10x+12$
8. $4y^4+32y^3+83y^2+76y+21=0$
9. $a^4-11a^3+44a^2-76a+48=0$
10. $\dfrac{x^2}{12}+\dfrac{1}{2x^2}+\dfrac{5}{12x}=\dfrac{1}{2x^2}+\frac{1}{2}$
11. $\frac{1}{4}+\dfrac{4}{x^2}=\dfrac{4x-1}{4x^2+12}$.

266. Show by the remainder theorem that the following have $a-b$ as a factor and find the other factor.

1. a^2-b^2
2. a^3-b^3
3. a^4-b^4
4. a^5-b^5
5. a^6-b^6
6. a^7-b^7
7. a^8-b^8
8. a^9-b^9.

9. Make a rule for forming the other factor in 1–8.

267. Show by the remainder theorem that the following have $a+b$ as a factor, and find the other factor.

1. a^3+b^3
2. a^5+b^5
3. a^7+b^7
4. a^9+b^9
5. $a^{11}+b^{11}$
6. a^2-b^2
7. a^4-b^4
8. a^6-b^6
9. a^8-b^8
10. $a^{10}-b^{10}$.

11. Make a rule for forming the quotient of the sum of the same odd powers of two numbers by the sum of those numbers.

12. Make a rule for forming the quotient of the difference of the same even powers of two numbers by the sum of those numbers.

Miscellaneous Exercises

268. Factor, and check the results:

1. m^2-n^2
2. $4a^2-9b^2$
3. $1-r^4$
4. x^2y-y^3
5. u^3+v^3
6. c^6-d^6
7. a^5+b^5
8. $450-2a^2$
9. g^6+h^6
10. a^9+b^9 (3 factors)
11. $x^{12}+y^{12}$
12. $m^{12}-n^{12}$
13. w^4+4
14. $a^4b^2-6a^3b+9a^2$

15. $x^4y^2+2x^2y^2z^2+y^2z^4$
16. $l^3mn^2-l^2mn^2x-l^2m^2n^2$
17. $y^2-ay-42a^2$
18. $5b^2+10b-15$
19. $3a^2-9ab-210b^2$
20. $9m^2-24mn+16n^2$
21. $4x^2+32xy+39y^2$
22. $bcy+acy^2+by^3+ay^4$
23. $m^2-6bm-4+9b^2$
24. $72r^2+41r-45$
25. x^3+x^2-3x-2
26. $6a^{2n}+a^nb^n-15b^{2n}$
27. $a^4b^2+a^3b^3-a^3b^2c$
28. $10am+45a^2b^2+25b^4$
29. $a^4+26a^2b^2+25b^4$
30. $30c^2-cd-20d^2$
31. $27+a^3b^6c^9$
32. a^5+1024
33. $10x^2r+33xr-7r$
34. $a^2+b^2-c^2-2ab$
35. a^2-b^2+a+b
36. $110-x-x^2$
37. $4x^4-61x^2y^2+9y^4$
38. $ac-ax-7bc+7bx$
39. $3m^3+5m^2-7m-1$
40. x^4-13x^2+36
41. $24x^2+37xy-72y^2$
42. $6(a+b)^2-(a+b)x-12x^2$
43. $65a^2c^2+8acd-d^2$
44. $m^2+6m-x^2+9-4xy-4y^2$
45. $12y^3+3y^2-8y-2$
46. $3x^3+2x^2-2x-1$
47. a^3+b^3+a+b
48. $4a^2b^2-(a^2+b^2-c^2)^2$
49. $9m^2-9a^2+64n^2-64b^2+48mn+48ab$
50. $x^2+9y^2+25z^2-6xy-10xz+30yz$.

269. Solve the following equations, and check results:

1. $ay+by-cm=ac+bc-my$. Solve for y
2. $2a^2r-10a^3b+55ab^2c^2=11bc^2r$. Solve for b
3. $t^2+24a^2=14at$. Solve for t; for a
4. $m^2=4m+1$. Solve for m
5. $16x+15=15x^2$. Solve for x
6. $n^3+60=4n^2+17n$. Solve for n
7. $ay=by+r+s$. Solve for y
8. $3k=5ky+3a-5ay+3b-5by$. Solve for k

9. $v^2+3v+7=0$. Solve for v

10. $w^2+8=0$. Solve for w

11. $\frac{2x^2}{x^2-1}+\frac{x}{x-1}=\frac{x}{x+1}+3$. Solve for x

12. $\frac{2x}{x-3y}+\frac{4}{x}=\frac{16}{x(x-3y)}$ and $\frac{x}{3}-y=0$. Find x and y

13. $\frac{2}{m}+n=4$, and $\frac{1}{m-1}+\frac{1}{n-2}=2$. Find m and n

14. $\frac{a}{mx}+\frac{b}{nx}+\frac{c}{rx}=d$. Find x

15. $\frac{ax}{a^2-b^2}-\frac{b}{nx}=a^2+b^2$

16. $\frac{bx}{a-b}=\frac{x+3b}{a+b}-1$

17. $\frac{5m^2+7m+4}{15m^2+m-6}=\frac{3m^2+6m+7}{9m^2+6m+3}$

18. $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$

19. $\frac{2x-1}{x^2+2}+\frac{3x^2+6}{2x-1}=4$

20. $\frac{x}{2}+\frac{2}{x^2+x}=2-\frac{x}{2}$.

270. Reduce to lowest terms:

1. $\frac{x^3-4x^2+x+6}{x^3-6x^2+11x-6}$

2. $\frac{ac-bc-ad+bd}{ac+ad-bc-bd}$

3. $\frac{2m^3+5m^2-12m}{7m^3+25m^2-12m}$

4. $\frac{a^2-2ab+b^2-c^2}{b^2-2bc+c^2-a^2}$

5. $\frac{x^3-5x^2y-2xy^2+24y^3}{x^3+3x^2y-10xy^2-24y^3}$.

271. Perform the indicated operations:

1. $\frac{x+6}{x-3}-\frac{x+4}{x-2}$

2. $\frac{a}{3a-3b}-\frac{b}{2b-2a}$

3. $\dfrac{m^2-9}{m^2+m-12}-\dfrac{x^2+x-20}{x^2-25}$

4. $\dfrac{y-3}{y+5}+\dfrac{y+3}{y-4}-\dfrac{y^2+2y+57}{y^2+y-20}$

5. $\dfrac{r^2+r-5}{2r^2-11r+12}-\dfrac{r^2+r-1}{2r^2+5r-12}$

6. $\dfrac{a^{3n}}{a^n-1}-\dfrac{a^{2n}}{a^n+1}-\dfrac{1}{a^n-1}+\dfrac{1}{a^n+1}$

7. $\dfrac{x^4-y^4}{x^3-y^3}\cdot\dfrac{x^2-y^2}{x^3+y^3}\cdot\dfrac{x^6-y^6}{x^2+y^2}$

8. $\dfrac{x^2-4x+3}{x^2+5x+6}\cdot\dfrac{x^2+8x+15}{x^2-5x+4}\div\dfrac{x^2+2x-15}{x^2-2x-8}$

9. $\left(m+\dfrac{rl}{m}\right)\left(r-\dfrac{m^2}{r}\right)$ 10. $\dfrac{ab}{b+a}\left(\dfrac{b}{a}-\dfrac{a}{b}\right)$

11. $\left(\dfrac{a}{a-x}-1\right)\div\left(1-\dfrac{a}{a+x}\right)$

12. $\dfrac{y^2-2ay+a^2}{y^2+4ay-5a^2}\cdot\dfrac{y^2-9a^2}{ay+2a^2}\cdot\dfrac{y^2+5ay}{y^2-4ay+3a}$

13. $\left(3m-5-\dfrac{2}{m}\right)\left(3m+5-\dfrac{2}{m}\right)\div\left(m-\dfrac{4}{m}\right)$

14. $\dfrac{1}{r-1}+\dfrac{r+1}{r^2-3r+2}+\dfrac{r-2}{2r^2-5r+2}$

15. $\dfrac{a^5-b^5}{a-b}\cdot\dfrac{a+b}{(a-b)^2}\cdot\dfrac{a^4-b^4}{(a+b)^3}\div\dfrac{a^2+b^2}{a^2-b^2}$.

272. Simplify:

$$\frac{\dfrac{x}{2}-\dfrac{y}{6}}{3x-\dfrac{y}{5}}$$

Multiply the numerator and denominator by 30, the common denominator of the fractions $\frac{x}{2}$, $\frac{y}{6}$, and $\frac{y}{5}$, whence the complex fraction reduces to $\frac{15x-5y}{18x-6y}=\frac{5}{6}$.

273. Simplify:

1. $\dfrac{\dfrac{1}{a}-\dfrac{2}{a^2}-\dfrac{3}{a^3}}{\dfrac{9}{a}-a}$

2. $\dfrac{x+\dfrac{6}{x-5}}{\dfrac{x-2}{x}}$

3. $\dfrac{b-1+\dfrac{6}{b-6}}{b-2+\dfrac{3}{b-6}}$

4. $\dfrac{\dfrac{y-2}{y^2-y-12}-\dfrac{y}{y^2+6y+9}}{\dfrac{6y^2+7y+2}{y^2-y-12}}$

5. $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}}\cdot\dfrac{m^2-n^2}{m^3+n^3}$

6. $\dfrac{\dfrac{x}{a}+\dfrac{a}{x}-2}{x-a}+\dfrac{\dfrac{x}{a}+\dfrac{a}{x}+2}{x+a}.$

274. Solve the following:

1. One farmer can plow a field in r days, and another can plow it in s days. In how many days can they plow it, both working together?

2. A can do a piece of work in f days, B in g days, C in h days. How many days will it take all together to do the work?

3. The height of a bin is 2 feet more than the width and one foot more than the length and the capacity is 336 cubic feet. Find the dimensions.

4. The ratio of the capacities of two bins is $a:b$, and together they hold a^3+b^3 bushels. What is the capacity of each?

5. A has m times as much money as B. Their total capital is invested in the following amounts: $(l+k)$ dollars, $(m+2k)$ dollars, and $(2k+km)$ dollars. How much money has each?

6. A quantity of flour was bought for \$96.00. If 8 barrels more had been obtained for the same amount of money, the

price per barrel would have been \$2.00 less. How many barrels were bought?

7. A and B are traveling in the same direction, and B is k yards in advance of A. If A travels at the rate of n yards per minute, and B at the rate of m yards per minute, in how many minutes will A overtake B?

8. N equal cubical boxes with edges equal to $n+3$ inches have a total capacity of 648 cubic inches. Find the number of boxes.

Summary

The **factors** of a number are the *exact divisors* of the number.

A **prime number** is a number that has no factors except *itself* and *unity*.

A **prime factor** is a factor that is a *prime number*.

An expression is equal to the *product* of all its *prime factors* for *all* values of its letters. Such equality is called identity.

An expression in which the *highest power* of the unknown is the *second power* is a **second degree expression,** or a **quadratic expression.**

An equation in which the *highest power* of the unknown is the *second power* is a **second degree equation,** or a **quadratic equation.** A quadratic equation is sometimes called a **quadratic.**

An expression of the form $ab+ac$ is factored thus:

$$ab+ac=a(b+c).$$

An expression of the form $am+bm+an+bn$ is factored thus:

$$am+bm+an+bn=(m+n)(a+b).$$

An expression of the form x^2+mx+n is factored thus:

$x^2+(a+b)x+ab=(x+a)(x+b)$, where $m=a+b$, and $n=ab$.

An expression of the form x^2-y^2 is factored thus:

$$x^2-y^2=(x+y)(x-y).$$

An expression of the form ax^2+bx+c is factored by finding a pair of factors of ax^2 and a pair of factors of c the sum of whose cross products is bx.

Numbers, such as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, etc., that can be expressed only approximately without the use of the radical sign are called **irrational numbers**. When expressed with the aid of the radical sign, irrational numbers are called **radicals**.

The product of two or more numbers is zero, if, and only if, at least one of the numbers is zero.

In division the dividend, D, the quotient, Q, the divisor, d, and the remainder, R, are connected by the relation

$$D=Q \cdot d+R.$$

The remainder, R, found by dividing an expression in x by $x-r$, equals the expression in r, i. e., the given expression with r put in place of x. This is called the **remainder theorem.**

CHAPTER XIV

POLYGONS. CONGRUENT TRIANGLES. RADICALS

Interior Angles of Polygons

275. A plane figure *bounded by straight lines* is a **polygon.**

1. Find the sum of the interior angles x, y, z, t, w, of a five-sided polygon (Fig. 242).

From a point P, inside the figure, draw lines to the vertices (Fig. 243).

How many triangles are thus formed?

What is the sum of the angles of each triangle? Of all the triangles?

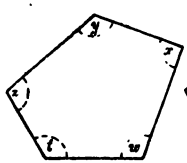


FIG. 242

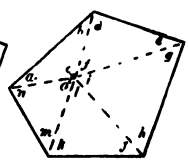


FIG. 243

What is the sum of the angles around P?

Prove that $t+w+x+y+z=5 \cdot 180-360$.

2. Prove that the sum of the interior angles

(1) of a 6-sided polygon is $6 \cdot 180-360$

(2) of a 7-sided polygon is $7 \cdot 180-360$

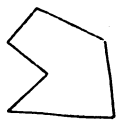
(3) of a 10-sided polygon is $10 \cdot 180-360$

(4) of an s-sided polygon is $s \cdot 180-360$

(5) of an n-sided polygon is $n \cdot 180-360$.

276. A **diagonal** of a polygon is a straight line joining two vertices that are not consecutive.

Convex Polygon

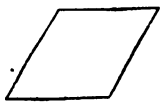

Concave Polygon

A polygon of three sides is a **triangle.**

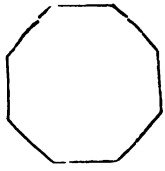
A polygon of four sides is a **quadrilateral.**

A polygon of five sides is a **pentagon.**
A polygon of six sides is a **hexagon.**
A polygon of seven sides is a **heptagon.**
A polygon of eight sides is an **octagon.**
A polygon of ten sides is a **decagon.**
A polygon of fifteen sides is **a pentedecagon.**
A polygon of n sides is an n-**gon.**

277. If all *sides* are *equal*, the polygon is **equilateral.** If

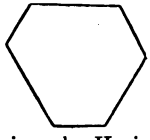


Equilateral Quadrilateral

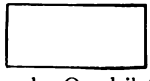


Equilateral Octagon

all *interior angles* are *equal*, the polygon is **equiangular.** A

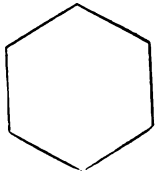


Equiangular Hexigon

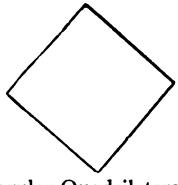


Equiangular Quadrilateral

polygon that is both *equilateral* and *equiangular* is a **regular polygon.**



Regular Hexagon



Regular Quadrilateral

278. THEOREM: *If d represents the number of degrees in the sum of the interior angles of a polygon, and n the number of sides, then*

$$d = n \cdot 180 - 360 \text{ (see §275).} \qquad (1)$$

1. Show that equation (1) may be written

$$d=(n-2)180. \qquad (2)$$

2. Letting r stand for the number of right angles in the sum of the interior angles of a polygon of n sides, show that equation (2) may be written

$$r=(n-2)2. \qquad (3)$$

3. Find from equations (2) and (3) the ratio, $d:r$.

4. Using equation (2) as a formula, find the sum of the interior angles of a quadrilateral; of a pentagon; of a hexagon; of a triangle.

5. Using equation (3) as a formula, find the sum of the interior angles of a heptagon; of an octagon; of a decagon; of a pentedecagon; of a polygon of 18 sides.

6. In the equation $d=(n-2)180$:

(1) What is the value of d, if n is 3? 9? 16? s?

(2) What is the value of n, if d is 900? 7200?

7. Solving for n the equation $d=(n-2)180$,

Show that $$n=\frac{d+360}{180}. \qquad (4)$$

8. By means of formula (4), find the number of sides of a polygon, the sum of whose interior angles is 3600°; 17100°; 180°.

9. Solving for n the equation $r=(n-2)2$,

Show that $$n=\frac{r+4}{2}. \qquad (5)$$

10. By means of formula (5), find the number of sides of a polygon, the sum of whose interior angles is 80 right angles; 192 right angles; 2 right angles.

11. Each interior angle of a regular pentagon is x. Find x in degrees.

In formula (2), p. 325, let $n=5$; then $d=5x$ (why?), and $5x=(5-2)180$. Solve for x.

12. Find the number of degrees in each interior angle of a regular hexagon; of a regular octagon; of a regular 17-gon; of a regular n-gon.

13. Using the equation obtained in the last part of problem 12 as a formula, find the number of degrees in each interior angle of a regular heptagon; of a regular decagon; of a regular pentedecagon.

14. An interior angle of a regular polygon is 120°. How many sides has the polygon?

In formula (2), p. 325, $d=120n$. Why?
Then $120n=(n-2)180$. Solve for n.

15. How many sides has a regular polygon, one of whose interior angles is 135°? 140°? 144°? 170°?

16. Can a tile floor be laid with tiles all having the shape of regular polygons of 3 sides? Of 4 sides? Of 5 sides? Of 6 sides? Of 8 sides? Of 15 sides?

17. In Fig. 244, show that $x=x'$.

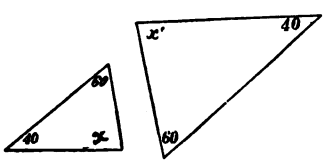


FIG. 244

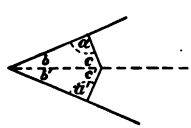


FIG. 245

18. In Fig. 245, $a=a'$, $b=b'$. Show that $c=c'$.

19. In Fig. 246, $a=a'$, $b=b'$. Show that $c=c'$.

20. In triangle A B C, Fig. 247, A O is perpendicular to B C, and $\angle B = \angle C$. Show that $y = y'$.

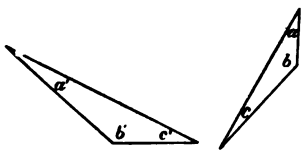


FIG. 246

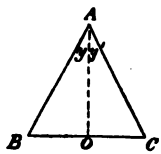


FIG. 247

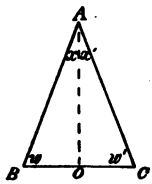


FIG. 248

21. In Fig. 248, $w = w'$, $x = x'$. Show that A O is perpendicular to B C.

If two angles of one triangle are equal to two angles of another, the third angle of the first triangle is equal to the third angle of the second.

Exterior Angles of Polygons

279. If one side of an n-gon is prolonged at each vertex, n exterior angles are formed, taking one at each vertex; for example, angles a, b, c, d, e, f (Fig. 249).

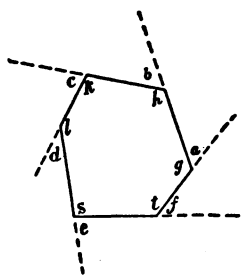


FIG. 249

1. Find the sum of the exterior angles of an n-gon.

The sum of the interior angle and the exterior angle at each vertex is 180°.

In Fig. 249, $a + g = 180$, $b + h = 180$, $c + k = 180$, etc.

Since there are n vertices, the sum of all the angles, interior and exterior, is n times 180 degrees, or $180n$ degrees.

$180n$ is the sum of all interior and exterior angles
$180n - 360$ is the sum of all interior angles—§ **278.**

Subtracting, 360 is the sum of all exterior angles.

2. Rotate a pencil through all the exterior angles of a polygon, taking one at each vertex. Through what part of a complete turn has the pencil rotated?

3. The sum of the interior angles of a polygon is 7 times the sum of the exterior angles. Find the number of sides.

Show that $(n-2)180=7\cdot 360$. Solve for n.

4. How many sides has a polygon in which the sum of the interior angles is 9 times the sum of the exterior angles?

5. How many sides has a polygon in which the sum of the interior angles is a times the sum of the exterior angles?

6. Each exterior angle of a regular polygon is $15°$. Find the number of sides.

7. Each exterior angle of a regular 8-gon is x. Find x in degrees.

8. Each exterior angle of a regular n-gon is y. Find y in degrees.

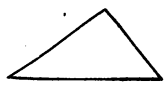
9. An exterior angle of a regular polygon is $\frac{1}{8}$ of the adjacent angle. Find the number of sides.

280. Some of the following definitions have been stated in preceding chapters; they are here given again for convenience.

TRIANGLES

281. An **isosceles triangle** is a triangle that has two equal sides.

Isosceles Triangle



Scalene Triangle

282. A **scalene triangle** is a triangle that has no two equal sides.

283. A **right triangle** is a triangle that has a right angle.

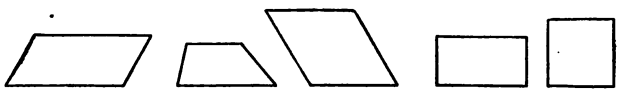
The side opposite the right angle is the *hypotenuse.*

The symbol for triangle is $\triangle$; for triangles is $\triangle s$.

Right Triangle

QUADRILATERALS

284. A **trapezoid** is a quadrilateral having *one* pair of parallel sides.



Parallelogram Trapezoid Rhombus Rectangle Square

285. A **parallelogram** is a quadrilateral having *two* pairs of parallel sides.

286. A **rhombus** is an *equilateral* quadrilateral.

287. A **rectangle** is an *equiangular* quadrilateral.

288. A **square** is a *regular* quadrilateral.

1. Show that a parallelogram is a trapezoid.
2. Show that a square is a rectangle.
3. Show that a square is a rhombus.
4. Is every trapezoid a parallelogram?
5. Is every rectangle a square?
6. Is every rhombus a square?

Congruent Triangles

289. Inaccessible distances frequently may be determined by triangles that are both equal and similar.

1. To find the distance from a point A to a point B on the opposite side of a river (Fig. 250) a surveyor lays off line C D making angle x equal to the angle x', and the line A D, making angle y equal to the angle y'.

What corresponding parts of triangles A B C and A D C are equal?

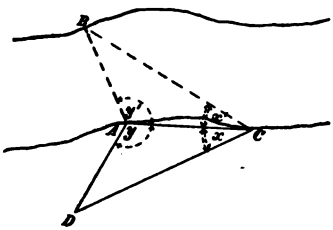


FIG. 250

How do triangles A B C and A D C seem to compare as to size and shape?

A D is measured and found to be 310 feet long. What seems to be the length of A B?

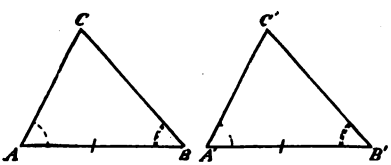


FIG. 251

2. Draw on paper a triangle as A B C (Fig. 251). Construct another triangle A′B′C′ as follows:

make A′B′ = A B
∠A′ = ∠A
∠B′ = ∠B.

Use the protractor, or the construction given in problem 24, pp. 148, 149.

Cut triangle A′B′C′ from the paper and see if it can be made to fit upon triangle A B C.

How do the triangles A B C and A′B′C′ compare as to size and shape?

Triangles which can be made to fit one over the other, or to coincide, are called *congruent* triangles.

If A′B′C′ can be made to fit on A B C, what must be true of the relations of the parts B C and B′C′? Of the parts C A and C′A′?

3. Draw the triangles of problem 2 on the blackboard. On thin paper make a trace of A′B′C′ with colored crayon, and fit the trace over A B C.

In congruent triangles the parts (the angles and the sides) which coincide, when the figures are made to coincide, are called *corresponding parts.*

Hence, it may be inferred that any part (angle or side) of one of two congruent triangles is equal to the corresponding part of the other.

In problem 2, triangle A′B′C′ was constructed so that one side and the two adjacent angles were equal to one side and the two adjacent angles of triangle A B C. It was seen that the two triangles can be made to coincide (are congruent).

4. It can be proved, without actually placing one triangle over the other, that *any two triangles* are congruent if a side and the two adjacent angles of one triangle are equal to a side and the two adjacent angles of the other.

The proof of this theorem may be reasoned out thus: Let A B C and A′B′C′ (Fig. 252) be any two triangles having

$$AB = A'B'$$
$$\angle A = \angle A'$$
$$\angle B = \angle B'.$$

Imagine triangle A B C placed on A′B′C′, the side A B coin-

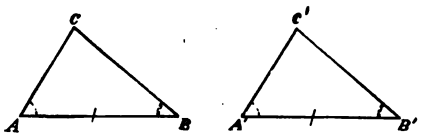


FIG. 252

ciding with side A′B′. (Why can A B be made to coincide with A′B′?)

Since $\angle A = \angle A'$, the side A C falls along A′C′ and the point C somewhere on A′C′.

Since $\angle B = \angle B'$, the side B C falls along B′C′ and the point C somewhere on B′C′.

Since C must be on both lines A′C′ and B′C′, it must fall on the point of intersection of A′C′ and B′C′. (Why?)

Therefore, C falls on C′.

Then the triangles A B C and A′B′C′ coincide throughout and are congruent.

In the same way in which A B C and A′B′C′ were made to coincide, any two triangles can be made to coincide if a side and two adjacent angles of one triangle are equal to a side and the two adjacent angles of the other.

290. THEOREM: *If in two triangles a side and the two adjacent angles of the one are equal respectively to a side and the two adjacent angles of the other, the triangles are congruent.*

1. PROVE: *In two right triangles if the hypotenuse and a side of the one are equal respectively to the hypotenuse and a side of the other, the triangles are congruent.*

Let the hypotenuse, c, and the side, a, of one triangle (Fig. 253) be equal respectively to the hypotenuse, c, and the side, a, of the other triangle.

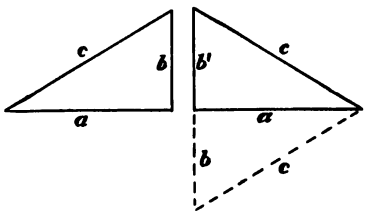


FIG. 253

To prove the triangles congruent.

Proof. Place the triangle at the left so that the equal sides, a, coincide, as shown.

The line $b'+b$ is straight. Why?

The figure formed of the two triangles is an isosceles triangle. Why?

Complete the proof.

2. Give the proof, using Fig. 254.

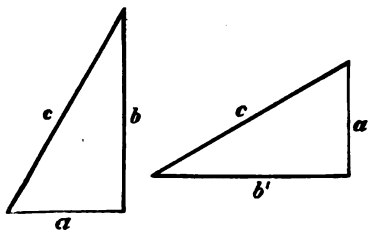


FIG. 254

3. In triangles R S T and X Y Z (Fig. 255), the parts marked in the same way are equal. Show, as in problem 4, p. 331, that triangles R S T and X Y Z are congruent.

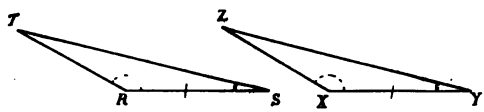


FIG. 255

How do the sides T R and X Z compare in length? T S and Z Y?

How do angles T and Z compare in size?

4. In triangle D E F (Fig. 256), E G bisects angle E and makes equal angles with D F (i. e, is perpendicular to D F). What parts of the two triangles are known to be equal? Show that triangles D E G and E F G are equal in all respects.

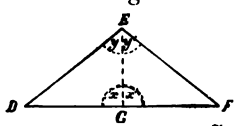


FIG. 256

Hence, side D E = side E F; side D G = side G F; and angle D = angle F.

5. In triangles A B C and A′B′C′ (Fig. 257) the parts marked in the same way are equal. Prove that C A = C′A′. What other parts are equal? Give reason.

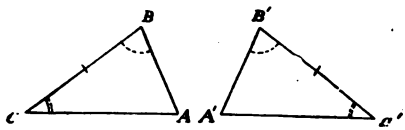


FIG. 257

6. In Fig. 258, A B is equal to B C, and the angles at A and C are equal as indicated. What other angles of the triangles A E B and B C D are equal? Prove the triangles congruent, and angle A E B equal to angle B D C.

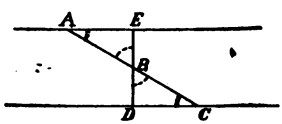


FIG. 258

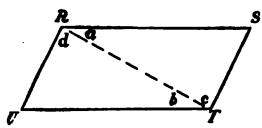


FIG. 259

7. In Fig. 259, $a=b$ and $c=d$. Prove, by means of congruent triangles, that R S = U T, and R U = S T.

8. In the parallellogram A B C D, Fig. 260, $x=x'$, $y=y'$, A B = D C. Prove that A O = O C, B O = O D.

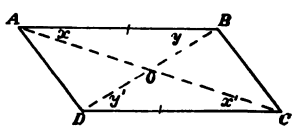


FIG. 260

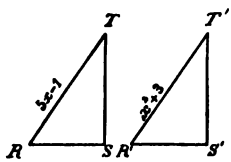


FIG. 261

9. Triangles R S T and R′S′T′ (Fig. 261) are congruent. Find x and the corresponding sides R T and R′T′.

10. The triangles in Fig. 262 are congruent. Find s and t. How long is B C? A B?

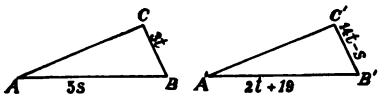


FIG. 262

11. Draw a triangle, as A B C (Fig. 263). Construct a triangle having the sides equal respectively to the sides of triangle A B C.

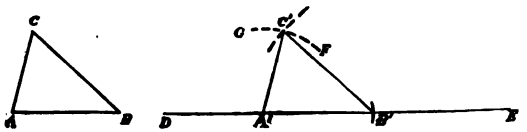


FIG. 263

On a line, as D E, lay off A′B′ equal in length to A B.

With A′ as center and a radius equal to A C, draw an arc F G above D E.

With B′ as center and radius equal to B C, draw an arc meeting arc F G at C′.

Draw lines A′C′ and B′C′.

Triangle A′B′C′ is the required triangle.

12. Prove that triangle A′B′C′, Fig. 263, is congruent to triangle A B C.

Proof. In triangle A B C (Fig. 263), with A as center and A C as radius, draw a semicircle x (see Fig. 264).

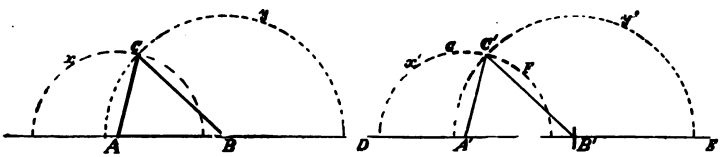


FIG. 264

In △A B C (Fig. 263), with B as center and B C as radius, draw a semicircle y (see Fig. 264).

In $\triangle A'B'C'$ (Fig. 263), extend arcs through C′, completing semicircles x' and y' (see Fig. 264).

Imagine triangle A B C placed on A′B′C′. Then A B can be made to coincide with A′B′. (Why?)

Since A C is equal to A′C′ (why?), semicircle x coincides with semicircle x', for circles drawn with the same center and radius coincide.

Therefore point C must fall somewhere on the semicircle x'. (Why?)

Since B C is equal to B′C′, semicircle y coincides with semicircle y', and point C lies somewhere on semicircle y'.

Therefore point C must be on the point of intersection of x' and y'.

Then A C coincides with A′C′, and B C with B′C′.

The triangles coincide throughout and are congruent.

13. Two triangles as in Fig. 265 or Fig. 266 have three sides of the one equal to the corresponding sides of the other.

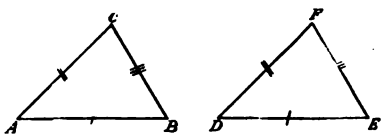


FIG. 265

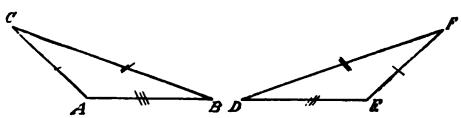


FIG. 266

Prove as in problem 12 that $\triangle$A B C and $\triangle$D E F are congruent.

291. Problems 11 and 12 establish the truth of the following theorem:

If two triangles have three sides of one equal to the three corresponding sides of the other, the triangles are congruent.

292. We have thus far proved three theorems on congruent triangles, viz.:

I. *If two sides and the included angle of one triangle are equal, each to each, to two sides and the included angle of another triangle, the triangles are congruent.* See §**145**, p. 202.

II. *If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of another, the triangles are congruent.* See §**290**, p. 332.

III. *If three sides of one triangle are equal, respectively, to the three sides of another, the triangles are congruent.* See §**291**, p. 336.

These three theorems on congruent triangles are used in proving other theorems in this chapter. The statements of the theorems should therefore be carefully memorized by the student.

293. Circles and Triangles

1. Prove the theorem: *If a point is on the perpendicular bisector of a line, it is equally distant from the extremities of the line.**

Let L M be perpendicular to A B at the middle point O, and P be any point on L M.

To prove P equally distant from A and B.

Proof. Draw P A and P B and prove triangles P O A and P O B congruent. (§**145**.)

Then P A = P B. Why?

But P is *any* point on A B. Therefore, any point on the perpendicular bisector of a line is equally distant from the extremities of the line.

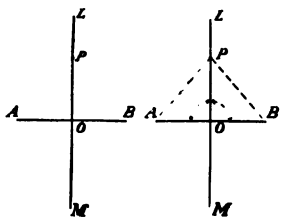


FIG. 267

* In the statement of this theorem, and of some others following, it is implied that there is only one perpendicular bisector of a line.

2. Prove the theorem in 1, using Fig. 268.

3. Prove the theorem: *If a point is equally distant from the extremities of a line, it is on the perpendicular bisector of the line.*

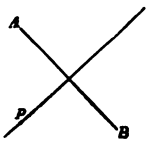


FIG. 268

Let P be any point equally distant from the extremities of the line A B; that is, P A = P B. (Fig. 269.)

To prove that P is on the perpendicular bisector of A B.

Proof. From O, the mid-point of A B, draw a line to P.

Prove triangles P O A and P O B congruent. (§291.)

Then $x=y$ and P O is perpendicular to A B. Why?

But P O also bisects A B. Why?

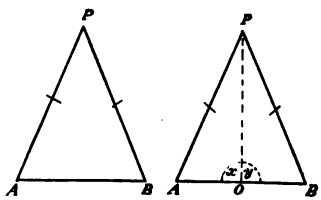


FIG. 269

Therefore P is on the perpendicular bisector of A B.

But P is *any* point equally distant from the extremities of A B.

Therefore, any point that is equally distant from the extremities of a line is on the perpendicular bisector of the line.

4. Prove the theorem in 3, using Fig. 270; Fig. 271.

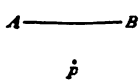


FIG. 270

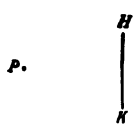


FIG. 271

5. Assuming that the perpendicular bisectors X M and Y R of two sides of triangle A B C meet at O, Fig. 272, prove

(1) O B = O C, O C = O A, and O B = O A,

(2) O is on the perpendicular bisector of A B.

Draw a circle with O as center, and O B as radius, and notice that the circle passes through C and A.

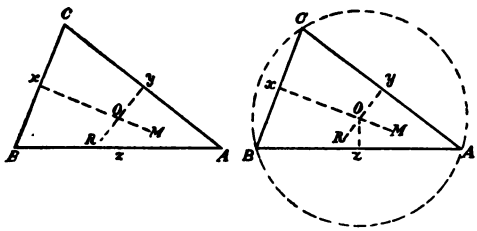


FIG. 272

A circle passing through the three vertices of a triangle is **circumscribed** about the triangle.

6. Draw a triangle having an obtuse angle. Circumscribe a circle about the triangle.

Construct the perpendicular bisectors of two sides (see problem 15, p. 147). Prove that the point where the perpendicular bisectors meet is equally distant from the three vertices.

Notice the position of the center of the circle with respect to the triangle.

7. Circumscribe a circle about a right triangle. Notice the position of the center of the circle with respect to the triangle.

In problem 5, where does the center of the circle lie with respect to the triangle?

294. The *distance* from a point to a line is the length of the *perpendicular* from the point to the line.

The symbol $\perp$ stands for "perpendicular," "perpendicular to," or "is perpendicular to."

The symbol $\perp^s$ stands for "perpendiculars."

1. Prove the theorem:

If a point is on the bisector of an angle it is equidistant from the sides of the angle.

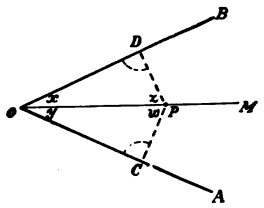


FIG. 273

Let M O be the bisector of A O B and P be any point on M O.

To prove P equidistant from A O and B O.

Proof. Draw P C ⊥ A O and P D ⊥ B O.

Prove triangles P O C and P O D congruent. (§**292.**)

Then P C = P D. Why?

2. Construct the bisector of an obtuse angle A O B. From any point P on the bisector, construct perpendiculars, P C and P D, to the sides. Draw a circle with P as center, that passes through C and D.

The sides A O and B O (Fig. 274) are *tangent* to the circle.

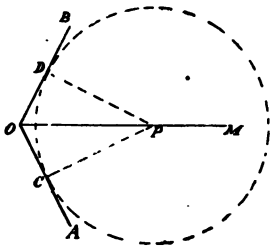


FIG. 274

3. Prove the theorem:

If a point is equidistant from the sides of an angle it is on the bisector of the angle.

Let P be any point equidistant from the sides of angle A O B; that is, P C ⊥ O A, and P D ⊥ O B, and P C = P D.

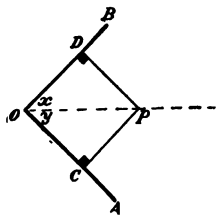


FIG. 275

To prove that P is on the bisector of angle A O B.

Proof. Draw a straight line through P and O. Prove triangles P O C and P O D congruent (problem 1, p. 332).

Then $x = y$ and P O bisects A O B. Why?

Therefore P is on the bisector of A O B.

4. The bisectors C M and A L of two angles of triangle

A B C meet at O. Perpendiculars are drawn from O to the sides.

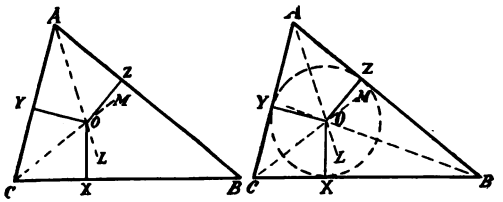


FIG. 276

Prove (1) O X = O Y, O Y = O Z, and O Z = O X.
(2) O is on the bisector of angle B.

Draw a circle with O as center, and O X as radius, and notice that the circle passes through Y and Z.

Polygons is a general name for triangles, quadrilaterals, pentagons, hexagons, etc. Polygon is another name for an n-gon.

A circle entirely within a polygon, all sides of which are tangent to the circle, is **inscribed in the polygon.**

5. Draw a triangle having an obtuse angle. Draw the inscribed circle. Construct the bisectors of two angles (see problem 20, p. 148). Prove that the point where the bisectors meet is equidistant from the three sides of the triangle.

6. Inscribe a circle in a right triangle.

7. In Fig. 277, show that

Area of B O C $= \frac{1}{2}r \cdot a$

Area of A O C $= \frac{1}{2}r \cdot b$

Area of A O B $= \frac{1}{2}r \cdot c$.

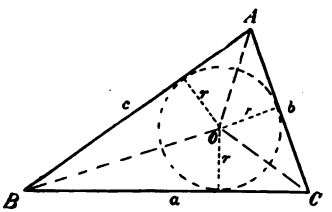


FIG. 277

Therefore, area of A B C $= \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc$

$$= \tfrac{1}{2}r(a+b+c)$$
$$= \tfrac{1}{2}(a+b+c)r.$$

8. Letting A stand for the area of triangle A B C, Fig. 277, and s stand for $\frac{1}{2}(a+b+c)$, show that

$$A = s \cdot r$$

$$\text{and } r = \frac{A}{s}.$$

THEOREM: *The radius of a circle inscribed in a triangle is equal to the area of the triangle divided by one-half of the perimeter.*

Application of the Theorems on Congruent Triangles

295. The theorems on congruent triangles §292 may be used in proving the correctness of the fundamental constructions in chap. vii, §111.

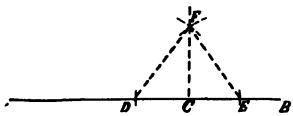


FIG. 278

1. Prove that C F (Fig. 174, p. 145) is perpendicular to A B.

Proof. Draw the lines D F and E F as in Fig 278.

Then in triangles D C F and E C F the following relations are known.

D C = C E Why?
E F = D F Why?
C F = C F.

Therefore triangles D C F and E D F are congruent. Why? $\angle$D C F = $\angle$E C F. Why? C F $\perp$ D E. Why?

Hence the line C F, if constructed as in §111, is perpendicular to A B at the point C.

The symbol $\therefore$ means "therefore," or "hence."

2. Prove that the construction in problem 8 (p. 146) makes C F perpendicular to A B.

Proof. In Fig. 176 draw D C and C E as in Fig. 279.

C is equally distant from D and E. Why?

∴ C lies on the perpendicular bisector of D E (see problem 3, p. 340).

Since F is equally distant from D and E (why?), F lies on the perpendicular bisector of D E. (Why?)

∴ the perpendicular bisector of D E contains C and F.

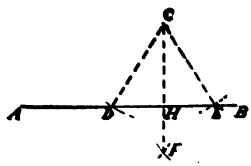


FIG. 279

∴ line C F must be the perpendicular bisector of D E, for through two points only one straight line can be drawn.

∴ C F is perpendicular to A B. Why?

3. Give proof for problem 15 (p. 147).

In Fig. 180, draw the lines A F, B F, A G, G B (see Fig. 280).

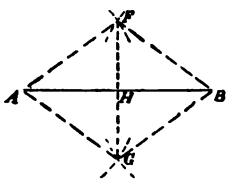


FIG. 280

Show as in problem 2 that F G is the perpendicular bisector of A B.

Then H is the midpoint of A B. Why?

4. Give proof for problem 19 (p. 147).

In Fig. 181, draw lines E F and D F. Prove that triangles B E F and B D F are congruent.

Then angle C B F equals angle A B F. Why?

5. Prove problem 24 (p. 148).

In Fig. 185, draw G H and O M.

6. In Fig. 281, A C = C B and $x = y$.

Prove $\angle A = \angle B$.

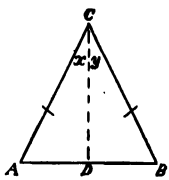


FIG. 281

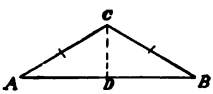


FIG. 282

Prove triangles A C D and B C D congruent (§ 145).

7. In triangle A B C, Fig. 282, A C = B C; prove $\angle A = \angle B$.

Bisect $\angle C$ (see § 111) and use problem 6.

THEOREM: *If a triangle has two equal sides, the angles of the triangle opposite the equal sides are equal.*

8. In Fig. 283, A C = C B. C D is drawn from the vertex to the midpoint of A B. Prove $x = y$

$z = u$

C D ⊥ A B.

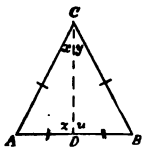


FIG. 283

THEOREM: *In an isosceles triangle the line joining the vertex to the midpoint of the base bisects the vertex-angle, and is perpendicular to the base.*

9. *If two angles of a triangle are equal, the sides opposite them are equal.*

In this statement, it is assumed that two angles of a triangle are equal. From this it is inferred that the sides opposite them are equal. It is to be proved that this is true.

Let A B C be a triangle having ∠A = ∠B.

To prove A C = B C.

Proof. Bisect ∠C.

∠A = ∠B Why?

$x = y$ Why?

$r = s$ Why?

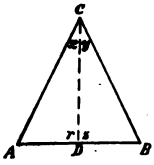


FIG. 284

Prove triangles A D C and B D C congruent.

Whence A C = B C. Why?

10. Prove that the line bisecting the angle at the vertex of an isosceles triangle bisects the base, and is perpendicular to the base.

What is assumed in this statement? What inference is drawn from this assumption?

11. Prove that the altitude of an isosceles triangle bisects the base and also the vertex-angle.

What is assumed here? What is the inference from the assumption?

12. PROVE: *A diagonal of a parallelogram divides it into two congruent triangles.*

What is the assumption? The inference from it?

Given a parallelogram A B C D, and B D one of the diagonals.

To prove that triangles A B D and C B D are congruent.

Proof. Since A B C D is a parallelogram A B is parallel to D C. (§285.)

$\therefore x = x'$ (see Summary, p. 151)

A D is parallel to B C. Why?

and $y = y'$. Why?

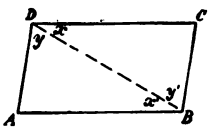


FIG. 285

$\therefore$ Triangles A B D and D B C are congruent. Why?

296. In every theorem one or more assumptions are made. From these assumptions, inferences (conclusions) are drawn. The assumptions in a theorem are called *the hypothesis* and the inferences *the conclusion.*

The hypothesis and conclusion together contain everything that is stated in the theorem. Nothing can be in the hypothesis or in the conclusion unless it is in the theorem.

1. State the hypothesis and conclusion in the following theorems:

(1) page 332, §290
(2) page 335, problem 16
(3) page 332, problem 5
(4) page 336, §291
(5) page 338, problem 3
(6) page 340, problem 3.

2. PROVE: *The opposite sides of a parallelogram are equal.*

Observe that the hypothesis in this theorem is: *A figure is a parallelogram;* the conclusion: *the opposite sides of this figure are equal.*

The hypothesis and conclusion must be stated always with reference to the particular figure used in the demonstration.

Hypothesis. A B C D is a parallelogram. (Fig. 286.)

Conclusion. A B = D C and A D = B C.

Proof. Draw A C.

Then triangle A B C is congruent to A D C. Why?

$\therefore$ A B = D C

and A D = B C. Why?

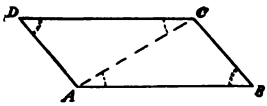


FIG. 286

The symbol $\cong$ may be used for "is congruent to" or "coincides with." The statement, *triangle A B C is congruent to triangle D E F*, may be written thus: $\triangle$A B C $\cong$ $\triangle$D E F.

Prove the following theorems:

3. *Parallel lines intercepted between parallel lines are equal* (Fig. 286).

4. *The diagonals of a parallelogram bisect each other.*

The proof for this theorem is suggested in problem 12, p. 334.

5. *A parallelogram having two adjacent sides equal is a rhombus.*

6. *The diagonals of a rhombus are perpendicular to each other.*

Hypothesis. A given figure is a rhombus, in particular A B C D, and A C and B D are the diagonals of A B C D.

Conclusion. The diagonals of the figure are perpendicular to each other, i. e., A C is perpendicular to B D.

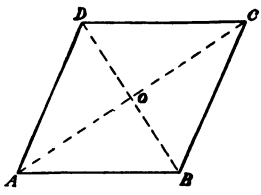


FIG. 287

Proof. D C = C B Why?

D O = O B Why?

Prove $\angle$D O C = $\angle$B O C.

Prove O C $\perp$ D B.

Then, A C $\perp$ D B.

7. *The diagonals of a rectangle are equal.*

In Fig. 288 prove A B C $\cong$ A B D (§287).

8. *If two adjacent sides and the included angle of one parallelogram are equal respectively to two adjacent sides and the included angle of another, the parallelograms are congruent.*

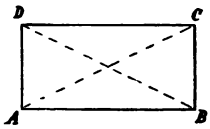


FIG. 288

In Fig. 289, $\triangle$I $\cong$ $\triangle$I′
$\triangle$II $\cong$ $\triangle$II′.

Show that parallelogram A B C D can be made to coincide with parallelogram A′B′C′D′.

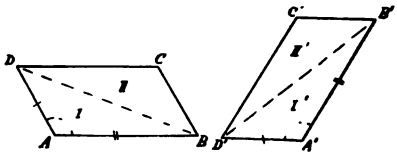


FIG. 289

Prove the following theorems:

9. *If two adjacent sides of one rectangle are equal respectively to two adjacent sides of another, the rectangles are congruent.*

10. *Two squares having a side of one equal to a side of the other are congruent.*

In §§105 and 106 the following theorem was studied: Two straight lines that make equal corresponding angles with a transversal are parallel.

Thus in Fig. 160, p. 140, if $a=e$, then A B || C D.*

11. Prove that if $c=e$ (Fig. 160), then $a=e$ and A B || C D.

12. Prove that if $c=e$, then $h=d$ and A B || C D.

* The symbol || means "is parallel to," or "are parallel."

13. Prove that if two lines make equal alternate interior angles with a transversal, the corresponding angles are equal.

14. Prove that if two lines are cut by a transversal making the sum of the interior angles on the same side equal to two right angles, the corresponding angles are equal.

In Fig. 160 let $d+e=180$. Then $d+a=180$. Therefore $a=e$.

15. Prove that two lines making equal alternate interior angles with a transversal are parallel.

16. Prove that if two lines are cut by a transversal making the sum of the interior angles on the same side equal to two right angles, the lines are parallel.

17. State three theorems which furnish tests for parallel lines.

18. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

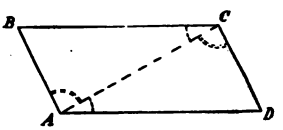


FIG. 290

Hypothesis. B C = A D. B C ∥ A D.
Conclusion. A B C D is a parallelogram.
Proof. Draw A C.
Then △A B C ≌ △A C D. Why?
∴ ∠B A C = ∠A C D.
∴ A B ∥ C D.
∴ A B C D is a parallelogram. Why?

19. *A rhombus is a parallelogram.* (§§285 and 286.)

20. *A rectangle is a parallelogram.* (§§285 and 287.)

21. *A square is a parallelogram.* (§§285 and 288.)

The Equilateral Triangle

297. The **altitude-lines** of a triangle are perpendiculars from the vertices to the opposite sides.

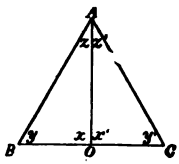


FIG. 291

1. Prove that an altitude-line of an equilateral triangle bisects the base.

Show by §292 that triangles A O B and A O C (Fig. 291) are congruent.

2. Prove that a line drawn from the vertex of an equilateral triangle to the mid-point of the opposite side, is perpendicular to that side.

Use §33, page 41.

The medians of an equilateral triangle are also altitude-lines.

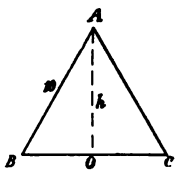


FIG. 292

3. Prove that the median to one side of an equilateral triangle bisects the angle opposite that side.

4. Find the altitude and the area of the equilateral triangle (Fig. 292).

5. Find the altitude and the area of an equilateral triangle whose side is 6; 8; 12; 3; 5.

Extracting Square Roots

298. The process of extracting square roots of numbers is used in finding approximate values of irrational numbers such as $\sqrt{75}$, $\sqrt{27}$, $\sqrt{\frac{27}{4}}$, $\sqrt{\frac{75}{4}}$, met with in the preceding problems 4 and 5.

1. Give the squares of the following numbers:

1, 2, 3, 4, 5, 6, 7, 8, 9
10, 20, 30, 40, 50, 60, 70, 80, 90
100, 200, 300, 400, 500, 600, 700, 800, 900.

2. Show that the square

(1) of any one-digit number has one or two digits

(2) of any two-digit number has three or four digits

(3) of any three-digit number has five or six digits.

3. Illustrate by squaring 456 that if a number, like 207,936, be marked off into two-digit groups, thus 20′79′36′, the number of groups is the same as the number of digits in the square root.

299. The numerals of arithmetic express numbers as so many tens and so many units, or so many hundreds, so many tens, and so many units, and so on.

To find the square root of a number means to find first the tens, then the units; or first the hundreds, then the tens, then the units, according as the root is a two-digit, or a three-digit number.

By problem 3 (§**298**) one can tell at the outset how many digits there are in the square root of a given number.

1. How many digits are there in the square roots of the following numbers?

1,681, 2,116, 961, 81, 121,104, 19,881, 964,324.

300. To understand the way to find square roots, it is necessary only to see how the tens and the units enter into the squares of numbers, and then, to learn how to reverse the order of steps.

ARITHMETICALLY	ALGEBRAICALLY
$57 = 50 + 7$	$n = t + u$
$57^2 = (50+7)^2$	$n^2 = (t+u)^2$
$= 50^2 + 2 \cdot 50 \cdot 7 + 7^2$	$= t^2 + 2tu + u^2$
$= 50^2 + (2 \cdot 50 + 7)7$	$= t^2 + (2t+u)u$
$= 2{,}500 + 700 + 49 = 3{,}249$	

REVERSING THE STEPS

		3,249 | 50+7 = 57, root
		2,500 = 50^2
trial divisor,	$2 \times 50 = 100$	749 = remainder
second root figure,	$7 = 7$	
	$2 \cdot 50 + 7 = 107$	749 = $(2 \cdot 50 + 7)7$
		0

$$
\begin{array}{lr|l}
 & & t^2+2tu+u^2 \;\big|\; t+u, \text{ root,} \\
 & & t^2 \qquad\qquad = \text{sq. of tens} \\
\hline
\text{Trial divisor,} & 2\times t = 2t & 2tu+u^2 = \text{remainder} \\
\text{Second root figure,} & u = u & \\
\hline
 & 2t+u & 2tu+u^2 = (2t+u)u \\
\hline
 & & 0
\end{array}
$$

1. Extract the square root of 183,184, omitting all unnecessary figures.

$$
\begin{array}{r|l}
 & 18'31'84' \;\big|\; \underline{428}, \text{ square root} \\
 & \underline{16} \\
80 & 231 \\
\underline{2} & \\
82 & \underline{164} \\
 & 6{,}784 \\
840 & \\
\underline{8} & \\
848 & \underline{6{,}784} \\
 & 0
\end{array}
$$

CHECK: $428^2 = 183{,}184$.

2. Extract the square roots of the following:

(1) 784	(5) 42,436	(9) 120,409
(2) 1,521	(6) 52,441	(10) 281,961
(3) 2,209	(7) 99,225	(11) 389,376
(4) 2,401	(8) 112,896	(12) 515,524.

301. Proceed with decimals precisely as with whole numbers, carrying the two-digit grouping toward the right of the decimal point, and bringing down always the next two figures of the given number.

1. Extract the square roots of the following:

19.36	1.5876	.0361
2.89	2.1316	51.2656
5.29	9.6721	.5329
114.49	11.0889	78.4996
210.64	.1225	.984064.

2. Find approximate values, to four decimal places, of the following:

$$\sqrt{2};\ \sqrt{3};\ \sqrt{5};\ \sqrt{6};\ \sqrt{7}.$$

Annex zeros and proceed as with decimals.

302. The square root of algebraic expressions is extracted in the same way as the square root of arithmetical numbers.

1. Extract the square root of

$$9x^2-30xz-12xy+4y^2+25z^2+20yz.$$

For convenience arrange the polynomial in the order of powers of x, y, and z thus:

		$9x^2-12xy+4y^2-30xz+20yz+25z^2$	square root $3x-2y-5z$
		$9x^2$	
1st trial divisor	$2\times 3x=6x$	$-12xy+4y^2$	
2d root number	$\dfrac{-12xy}{6x}=-2y$		
1st complete divisor	$6x-2y$	$-12xy+4y^2=$	$-2y\cdot(6x-2y)$
		$0-30xz+20yz+25z^2$	
2d trial divisor	$2(3x-2y)=6x-4y$		
3d root number	$\dfrac{-30xz}{6x}=-5z$		
2d complete divisor	$6x-4y-5z$	$-30xz+20yz+25z^2=$	$-5z\cdot(6x-4y-5z)$
		0	

CHECK: $(3x-2y-5z)^2=9x^2-12xy+4y^2-30xz+20yz+25z^2$.

2. Extract the square roots of the following:

(1) $25r^2-70rs+49s^2$

(2) $x^4+13x^2-12x-6x^3+4$

(3) $x^2+4xy+6xz+4y^2+12yz+9z^2$

(4) $9a^2-6ab^2-6ac+b^4+2b^2c+c^2$

(5) $x^4+2x+2x^3+3x^2+1$

(6) $1+a^6+a^2+2a-2a^3-2a^4$

(7) $9z^2-12az+4a^2+c^2+6cz-4ac$

(8) $r^2+25+r^4-2r^3+10r^2-10r$

(9) $s^6-2r^3s^3+2s^3-2r^3+r^6+1$

(10) $4x^{10}-4x^5y+4x^5z^3+y^2+z^6-2yz^3$.

303. Frequently the work of approximating the roots of numbers that are not squares is much shortened by simplifying the irrational numbers before extracting the roots.

1. In an equilateral triangle with side s and altitude h show that

$$\text{if } s=4, \text{ then } h=\sqrt{12}$$
$$\text{if } s=10, \text{ then } h=\sqrt{75}$$
$$\text{if } s=8, \text{ then } h=\sqrt{48}$$
$$\text{if } s=14, \text{ then } h=\sqrt{147}.$$

Instead of approximating directly the values of $\sqrt{12}$, $\sqrt{75}$, $\sqrt{48}$, $\sqrt{147}$, we may approximate directly only $\sqrt{3}$, and multiply the result by 2, 5, 4, 7, respectively, thus:

$$\sqrt{12}=2\sqrt{3}=2\times1.732+=3.464+$$
$$\sqrt{75}=5\sqrt{3}=5\times1.732+=8.660+$$
$$\sqrt{48}=4\sqrt{3}=4\times1.732+=6.928+$$
$$\sqrt{147}=7\sqrt{3}=7\times1.732+=12.124+.$$

304. In reducing the irrational numbers $\sqrt{12}$, $\sqrt{75}$, etc., in §**303**, the following principle is used:

The square root of a product is equal to the product of the square roots of the factors.

1. Prove, by multiplying, that $\sqrt{3^2\cdot5}=3\cdot\sqrt{5}$.

$3\cdot\sqrt{5}\cdot3\cdot\sqrt{5}=3\cdot3\cdot\sqrt{5}\cdot\sqrt{5}=3^2\cdot5$. Why?

2. Prove the following by multiplying:

(1) $\sqrt{2^2\cdot5^2}=2\cdot5$

(2) $\sqrt{x^2y^2}=xy$

(3) $\sqrt{a^2b^2}=ab$

(4) $\sqrt{7^2\cdot5}=7\cdot\sqrt{5}$

(5) $\sqrt{4\cdot25}=2\cdot5$

(6) $\sqrt{4\cdot7}=2\sqrt{7}$

(7) $\sqrt{16 \cdot 3} = 4 \cdot \sqrt{3}$

(8) $\sqrt{12} = 2 \cdot \sqrt{3}$

(9) $\sqrt{27} = 3\sqrt{3}$

(10) $\sqrt{75} = 5\sqrt{3}$

(11) $\sqrt{108} = 6\sqrt{3}$

(12) $\sqrt{a^2b^2c} = ab\sqrt{c}$.

3. Reduce $\sqrt{18}$.

$$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}.$$

4. Reduce $\sqrt{32a^3b^5x^3}$.

$$\sqrt{32a^3b^5x^3} = \sqrt{16a^2b^4x^2 \cdot 2abx} = \sqrt{16a^2b^4x^2} \cdot \sqrt{2abx} = 4abx\sqrt{2abx}.$$

5. Reduce the following irrational numbers, using the principle in §304:

(1) $\sqrt{45}$

(2) $\sqrt{50}$

(3) $\sqrt{28}$

(4) $\sqrt{48}$

(5) $\sqrt{125}$

(6) $\sqrt{200}$

(7) $\sqrt{288}$

(8) $\sqrt{128x^3y}$

(9) $3\sqrt{75x^5y^{10}}$

(10) $\sqrt{7a^2 - 14ab + 7b^2}$

(11) $\sqrt{3x^3(a-b)^5}$

(12) $\sqrt{27a^2 - 18b^2}$

(13) $\sqrt{3x^2(s^4 - 4s^2t^2 + 4t^4)}$

(14) $\sqrt{5 \cdot \frac{a^2}{b^2}}$.

305. It should be kept in mind that the square root of a sum is not equal to the sum of the square roots of the separate terms.

1. Prove, by multiplying, that $\sqrt{a^2+b^2}$ is not equal to $a+b$. Show that $(a+b)(a+b)$ is not equal to a^2+b^2.

2. Prove the following, by multiplying:

(1) $\sqrt{x^2-y^2}$ is not equal to $x-y$

(2) $\sqrt{9+16}$ is not equal to $3+4$

(3) $\sqrt{a^2-b}$ is not equal to $a-\sqrt{b}$.

3. Find an approximate value of the side s, and the area, of the equilateral triangle (Fig. 293).

In triangle A O B

$$s^2 = \frac{s^2}{4} + 36. \quad \text{Why?}$$

Show that $s = 6.928+$

and area A B C $= 20.784+$.

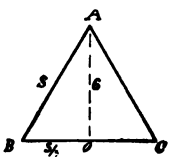


FIG. 293

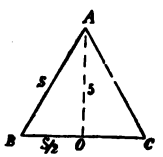


FIG. 294

4. Find an approximate value of a side, and the area, of an equilateral triangle having an altitude 3; 9; 12; 15; 18.

5. Find an approximate value of a side, and the area, of the equilateral triangle (Fig. 294).

Show that $s = \sqrt{\frac{100}{3}}$

$\sqrt{\frac{100}{3}} = \sqrt{100} \cdot \sqrt{\frac{1}{3}} = 10\sqrt{\frac{1}{3}}$. Why?

It is shown in the following problems 6 and 7, that

$$\sqrt{\tfrac{1}{3}} = \sqrt{1} \div \sqrt{3}.$$

Then $s = 10(1 \div 1.732+) = 5.773+$

and area A B C $= 14.430+$.

6. Prove the following by multiplication:

(1) $(\frac{2}{5})^2 = \frac{4}{25}$ (5) $(\frac{10}{12})^2 = \frac{100}{144}$

(2) $(\frac{4}{5})^2 = \frac{16}{25}$ (6) $(\frac{1}{13})^2 = \frac{1}{169}$

(3) $(\frac{7}{9})^2 = \frac{49}{81}$ (7) $(\frac{5}{3})^2 = \frac{25}{9}$

(4) $(\frac{3}{10})^2 = \frac{9}{100}$ (8) $(\frac{8}{3})^2 = \frac{64}{9}$.

7. Give the values of the following square roots:

(1) $\sqrt{\frac{4}{25}}$ (5) $\sqrt{\frac{81}{100}}$ (9) $\sqrt{\frac{100}{169}}$

(2) $\sqrt{\frac{16}{25}}$ (6) $\sqrt{\frac{49}{121}}$ (10) $\sqrt{\frac{121}{256}}$

(3) $\sqrt{\frac{49}{81}}$ (7) $\sqrt{\frac{64}{121}}$ (11) $\sqrt{\frac{169}{324}}$

(4) $\sqrt{\frac{9}{100}}$ (8) $\sqrt{\frac{36}{49}}$ (12) $\sqrt{\frac{361}{625}}$.

306. It is clear from 6 and 7, p. 355, that the square root of a common fraction is the square root of the numerator divided by the square root of the denominator.

1. Give the following square roots:

(1) $\sqrt{\frac{4}{a^2}}$

(2) $\sqrt{\frac{a^2}{b^2}}$

(3) $\sqrt{\frac{a^2}{b^2c^2}}$

(4) $\sqrt{\frac{a^2x^2}{b^2y^2}}$

(5) $\sqrt{\frac{16}{(m-n)^2}}$

(6) $\sqrt{\frac{9(a-b)^2}{25(a+b)^2}}$

(7) $\sqrt{\frac{a^2-2ax+x^2}{c^2+2cy+y^2}}$

(8) $\sqrt{\frac{m^2+2mn+n^2}{100(a+b)^2}}$

(9) $\sqrt{\frac{4x^2-4xy+y^2}{9y^2+6yz+z^2}}$

(10) $\sqrt{\frac{144(c+3d)^2}{16(4x^2-12xy+9y^2)}}$

(11) $\sqrt{\frac{16a^2b^4c^2(m-n)^4}{81c^6d^2(r+s)^6}}$

(12) $\sqrt{\frac{(6x-5y)^6(a+3d)^4}{(3a-c)^4(2r-5s)^2}}$.

2. Find approximate values, to four decimal places, of the following:

$$\sqrt{\tfrac{1}{3}};\ \sqrt{\tfrac{2}{3}};\ \sqrt{\tfrac{3}{5}};\ \sqrt{\tfrac{5}{7}}.$$

307. To find an approximate value of $\sqrt{\frac{2}{3}}$ we may approximate the square root of 2, obtaining 1.4142, and the square root of 3, obtaining 1.7320. Divide 1.4142 by 1.7320, obtaining .8165.

But the following method of finding an approximate value of $\sqrt{\frac{2}{3}}$ is shorter and gives a safer approximation:

$$\sqrt{\tfrac{2}{3}}=\sqrt{\tfrac{2}{3}\cdot\tfrac{3}{3}}=\sqrt{\tfrac{6}{9}}. \quad \text{Why?}$$

$$\sqrt{\tfrac{6}{9}}=\sqrt{\tfrac{1}{9}\cdot 6}=\tfrac{1}{3}\sqrt{6}. \quad \text{Why?}$$

$$\therefore \sqrt{\tfrac{2}{3}}=\tfrac{1}{3}\times 2.4492+=.8164+.$$

By the first method we obtain the square roots of two numbers, and then divide the root of the numerator by that of the denominator.

By the second method, we reduce the radical number first, then find the square root of only one number, and divide by 3.

1. Find an approximate value of $\sqrt{\frac{1}{3}}$.

$$\sqrt{\tfrac{1}{3}}=\sqrt{\tfrac{1}{3}\cdot\tfrac{3}{3}}=\sqrt{\tfrac{1}{9}\cdot 3}=\tfrac{1}{3}\sqrt{3}=.5773+.$$

2. Find an approximate value of $\sqrt{\frac{2a^4}{9x}}$.

$$\sqrt{\frac{2a^4}{9x}}=\sqrt{\frac{2a^4}{9x}\cdot\frac{x}{x}}=\sqrt{\frac{a^4}{9x^2}\cdot 2x}=\frac{a^2}{3x}\cdot\sqrt{2}\cdot\sqrt{x}=\frac{.4714a^2}{x}\sqrt{x},$$

approximately.

3. State a rule for simplifying irrational numbers like those in problems 1 and 2.

308. To simplify an irrational number which is the square root of a common fraction whose denominator is not a square:

(1) Multiply denominator and numerator by a number that will make the denominator of the resulting fraction a square.

(2) Factor and reduce to simplest form as in §**307**.

Simplify the following, and find approximate values to four decimal places:

1. $\sqrt{\frac{3}{5}}$
2. $\sqrt{\frac{3}{4}}$
3. $\sqrt{\frac{7}{a^2}}$
4. $\sqrt{\frac{6}{b}}$
5. $\sqrt{\frac{1}{2}}$
6. $\sqrt{\frac{1}{3}}$
7. $\sqrt{\frac{16}{5}}$
8. $\sqrt{\frac{7}{8}}$
9. $\sqrt{\frac{1}{7}}$
10. $\sqrt{\frac{25}{6}}$
11. $\sqrt{\frac{12}{7}}$
12. $\sqrt{1\frac{3}{5}}$
13. $\sqrt{\frac{3x^2yz^2}{a^3b}}$
14. $\sqrt{\frac{3}{a+b}}$
15. $\sqrt{\frac{2}{c(x+y)}}$.

16. Find the side and area of an equilateral triangle having an altitude 7; 2; 8; 10; 22.

17. In the equilateral triangle, Fig. 295, show that

$$s=\tfrac{2}{3}h\sqrt{3} \qquad (1)$$

$$\text{and area A B C}=\tfrac{1}{3}h^2\sqrt{3}. \qquad (2)$$

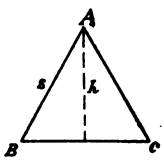


FIG. 295

18. Using equations (1) and (2), problem 17, as formulas find the side and the area of an equilateral triangle having an, altitude 6; 9; 12; 10; 14.

19. In the equilateral triangle, Fig. 295, show that

$$h=\frac{s}{2}\sqrt{3} \qquad (3)$$

$$\text{and area A B C}=\frac{s^2}{4}\sqrt{3}. \qquad (4)$$

20. By means of formulas (3) and (4), find the altitude and the area of an equilateral triangle having a side 6; 10; 9; 7.

21. In the equation $h=\frac{s}{2}\sqrt{3}$, substitute $\sqrt{5}$ for h, and show that $s=\frac{2\sqrt{5}}{\sqrt{3}}$.

To find an approximate value of $\frac{2\sqrt{5}}{\sqrt{3}}$, we may find an approximate square root of 5, obtaining 2.2360, and of 3, obtaining 1.7320. Multiply 2.2360 by 2, and divide the result by 1.7320, obtaining 2.5814.

The following method of finding an approximate value of $\frac{2\sqrt{5}}{\sqrt{3}}$ is shorter, and gives a safer approximation:

$\frac{2\sqrt{5}}{\sqrt{3}}=\frac{2\sqrt{5}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{5}\cdot\sqrt{3}}{3}=\frac{2\sqrt{15}}{3}=\frac{2}{3}\sqrt{15}$. Give reasons.

$$\sqrt{15}=3.8729+$$
$$\therefore \tfrac{2}{3}\sqrt{15}=2.5818+.$$

22. In the following, multiply numerator and denominator by a number that will make the denominator of the resulting fraction a rational number. Simplify and find approximate values to four decimal places:

(1) $\dfrac{3}{\sqrt{7}}$	(4) $\dfrac{\sqrt{7}}{\sqrt{18}}$	(7) $\dfrac{a^2b\sqrt{3}}{5\sqrt{2}}$
(2) $\dfrac{1}{2\sqrt{3}}$	(5) $\dfrac{1}{\sqrt{27}}$	(8) $\dfrac{4b}{a^3\sqrt{8}}$
(3) $\dfrac{5}{2\sqrt{12}}$	(6) $\dfrac{7\sqrt{11}}{\sqrt{10}}$	(9) $\dfrac{8a\sqrt{2a}}{\sqrt{32a}}$.

23. Solve the equation

$$h=\frac{s}{2}\sqrt{3},$$

for s in terms of h, and find the value of s for $h=\sqrt{10}$.

24. Solve the equation

$$s=\tfrac{2}{3}h\sqrt{3}$$

for h in terms of s, and find the value of h for $s=\sqrt{7}$.

Problems Involving Radicals

1. Find the area A of an equilateral triangle in terms of the side s.

2. Using the equation obtained in problem 1, find the side of an equilateral triangle whose area is 25 sq. in.; $16\sqrt{3}$ sq. in.; A square inches.

3. Find the area A of an equilateral triangle in terms of the altitude h.

4. Using the equation obtained in problem 3, find the altitude of an equilateral triangle whose area is $17\sqrt{3}$ sq. ft.; 35 sq. ft.; A sq. ft.

5. A plot of ground is to be staked off in the form of an equilateral triangle covering an area of 100 sq. ft. How long must the side be? The altitude?

6. How many degrees are there in each interior angle of an equilateral triangle?

Show that six equal equilateral triangles may be so placed in a plane about a point that they just fill the angular magnitude about the point.

How many degrees are there in each interior angle A, B, C, etc., of the polygon thus formed (Fig. 296)?

Show that the polygon is a regular hexagon.

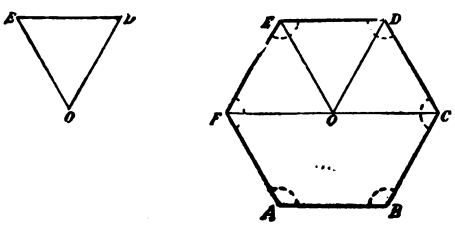


FIG. 296

7. Draw a regular hexagon starting with an equilateral triangle having a side 3. Circumscribe a circle about the hexagon. How long is the radius of the circle?

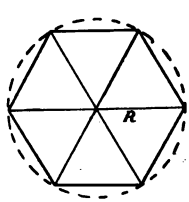


FIG. 297

8. Letting A be the area of a regular hexagon, and R the radius of the circumscribed circle (Fig. 297), show that a side of the regular hexagon is R, and that $A = \frac{3R}{2}\sqrt{3}$.

9. Find the area of a regular hexagon whose side is 2; 1; 3; 4; 15; s.

10. Find the radius of a circle circumscribed about a regular hexagon whose area is $3\sqrt{3}$; $6\sqrt{15}$; $9\sqrt{7}$; A.

11. Letting r be the radius of a circle inscribed in a regular hexagon (Fig. 298), find the area A of the hexagon, and s, the length of a side in terms of r.

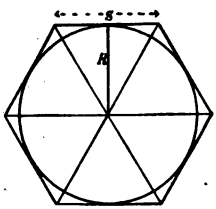


FIG. 298

12. Find the value of A and r, Fig. 298, in terms of s.

13. Make problems that can be solved by the aid of the formulas obtained in problems 11 and 12.

14. Show that a side s of a square inscribed in a circle (Fig. 299) is equal to $r\sqrt{2}$, and that the area A is equal to $2r^2$.

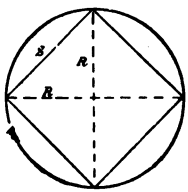


FIG. 299

15. Find the values of A and R (Fig. 299) in terms of S.

16. Make problems that can be solved by means of the formulas obtained in problems 14 and 15.

17. Calculate the radius of the circle circumscribed about a square whose area is 625; 200; y^2; A.

18. The diagonal of a square is 2 inches longer than a side x. Find the area.

19. The diagonal of a square is b inches longer than a side. Find the length of a side.

20. Show that the area of triangle A B C in the rhombus, Fig. 300, is equal to xy.

See problems 4 and 6, p. 346, and problem 19, p. 348.

21. Show that the area of the rhombus, Fig. 300, is $2xy$.

The area of a rhombus is one-half the product of the diagonals.

22. A side of a rhombus is 15, and one diagonal is 24. Find the other diagonal and the area.

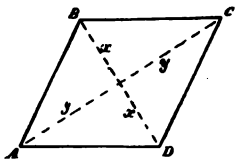


FIG. 300

23. A side of a rhombus is 12, and one diagonal is 18. Find the area.

24. One diagonal of a rhombus is twice as long as the other, and the area is 128. Find the length of each diagonal.

25. Find the area of an isosceles right triangle, the hypotenuse being 15; $3\sqrt{7}$; h.

Algebraic Problems Based on Geometry

1. The sides and angles of the triangles being designated as in Fig. 301, find c, x, and y.

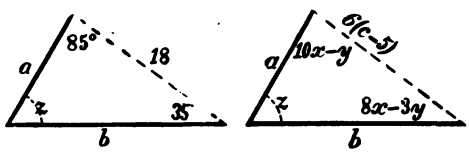


FIG. 301

2. In the two triangles of Fig. 302, $a=a'$, $c=c'$, and

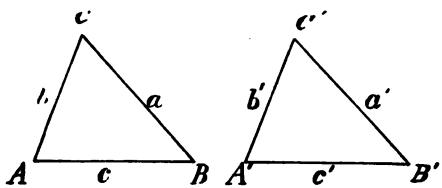


FIG. 302

$\angle B = \angle B'$; find the values of x, y, and z and of b, $\angle A$ and $\angle C$:

(1) If $b=3y+17$, $b'=73-2y$; $\angle A=5x-12°$, $\angle A'=60°-x$; $\angle C=150°-2z$, and $\angle C'=2(3z-5°)$.

(2) If $b=y+\frac{y}{2}$, $b'=\frac{3x}{5}+9$; $\angle A=3(4x+17z)$, $\angle A'=57°-15x$; $\angle C=23x-3z$, $\angle C'=8z+103°$.

(3) If $b=y(y-1)$, $b'=3(y-7)$; $\angle A=x(x-2°)$, $\angle A'=5(2x-7°)$; $\angle C=z(z-4°)$, $\angle C'=16(z-6°)$.

3. With angles and sides of two triangles of values that may be designated as in Fig. 303, find the values of a, c, and x.

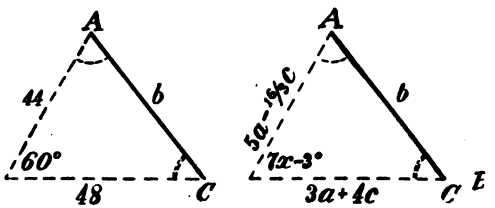


FIG. 303

4. In Fig. 304 find r and s.

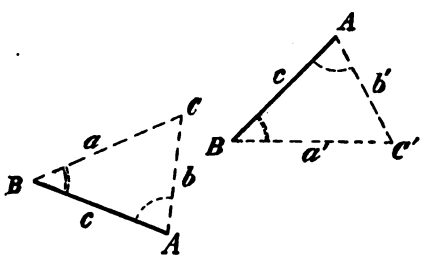


FIG. 304

(1) if $b=28$, $b'=2r-\frac{4s}{5}$, $a=23°$, and $a'=r+\frac{7s}{5}$;

(2) if $b=3r+2s-1$, $b'=68$, and $a=2r+2\cdot 5s$, $a'=60$;

(3) if $b=6r-s-6$, $b'=3(r+s)+2$, and $a=3r+2s+2$, and $a'=5r-\frac{2s}{3}-1$.

5. In Fig. 305 the triangles have sides and angles of values as designated. Find x, y, and z.

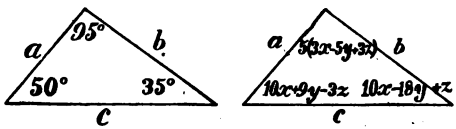


FIG. 305

6. The triangles in Fig. 306 have sides and angles of values as designated. Find x, y, and z.

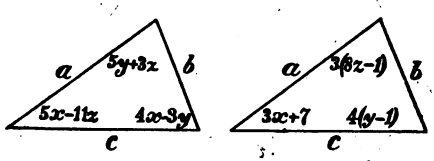


FIG. 306

7. The two triangles of Fig. 307 are congruent. If the values of the angles may be designated as shown, what are the values of x, y, and z?

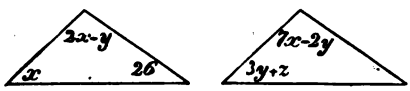


FIG. 307

8. The lengths of one pair of opposite sides of a parallelogram are $8x+6y$ and $12-y$, and the lengths of the other pair of opposite sides are $\frac{2x+y}{3}$ and $1-\frac{x+2y}{4}$. Find x, y, and the lengths of the sides of the parallelogram.

309. The corresponding sides of triangles whose angles are equal, each to each, are proportional.

For example, if in Fig. 308, $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, then $a:a' = b:b' = c:c'$.

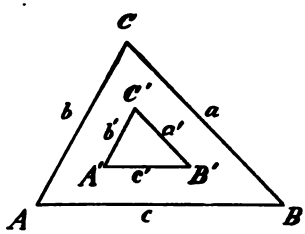


FIG. 308

The *ratio of any pair* of corresponding sides, as $a:a'$, $b:b'$, etc., is called the **ratio of similarity** of the figures.

If the ratio of similarity is 3, find the values of x, y, and z under the following conditions:

1. $a = x + z$, and $a' = 10 + \frac{y}{3}$
   $b = 3x - z$ and $b' = 4 + \frac{x}{3}$
   $c = 7z + 2x$ and $c' = \frac{140 + y}{3}$
2. $a = x + y$ and $a' = 5 - x$
   $b = 2x + y$ and $b' = 7 - 2z$
   $c = 3z + 2y$ and $c' = x + 1$
3. $a = x + y + 3z$ and $a' = 4$
   $b = 2x + y + 6z$ and $b' = 7$
   $c = x + 2y + 9z$ and $c' = 9$
4. $a = x + y + z$ and $a' = 3$
   $b = 3x + 4y - 8z$ and $b' = 3$
   $c = 4x + y - 2z$ and $c' = 4$
5. $a = 3x + y + 4z$ and $a' = 7$
   $b = x + 4y + 2z$ and $b' = 5$
   $c = x + 5y + 3z$ and $c' = 6$.